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INHALTSVERZEICHNIS

Ein axiomatischer Zugang zu einigen Winkelsätzen der ebenen Geometrie

Rudolf Fritsch

-1-

Epimorphisms of Separated Superconvex Spaces

Ralf Kemper

-18-

p-Banach Spaces and p-Totally Convex Spaces I

Ralf Kemper

-33-

p-Banach Spaces and p-Totally Convex Spaces II

Ralf Kemper

-55-

Positively Convex Spaces

Ralf Kemper

-74-

Strong Orthogonality, almost Membership, and Conservation of Invariance and Multiplicativity in Connection with stable Sets of Measures

D. Plachky

-95-

On the complete r-partite graphs and their line graphs

Peter Zörnig

-101-

Ein axiomatischer Zugang zu einigen Winkelsätzen der ebenen Geometrie

Rudolf Fritsch

Im 4. vorchristlichen Jahrhundert verfaßte Euklid die erste als wissenschaftlich anzusprechende Monographie mathematischen Inhalts [5]. Insbesondere sein Parallelenaxiom beschäftigte über Jahrtausende die Mathematiker, die die innere Abhängigkeit der Aussagen der euklidischen Geometrie untersuchten, um die Abhängigkeit oder Unabhängigkeit dieses Axioms von den übrigen Axiomen Euklids festzustellen. Dieses spezielle Problem wurde zwar durch C. F. Gauß (1777–1855), N. J. Lobatschevsky (1792–1856) und J. Bolyai (1802–1860) im 19. Jahrhundert gelöst, die die Unabhängigkeit des Parallelenaxioms nachweisen konnten (siehe [12, Seite 6]), aber das Bestreben der Mathematiker nach Klärung von inneren Abhängigkeiten blieb weiter bestehen. Einen kräftigen Aufschwung bekam diese Richtung mathematischer Forschung durch den aus Ostpreußen stammenden David Hilbert (1862–1943), der zu den bedeutendsten Mathematikern des 19. und 20. Jahrhunderts gehört. In seinem 1899 erstmalig erschienen Buch *Grundlagen der Geometrie* [6] stellt er nicht nur ein in sich geschlossenes Axiomensystem dar, sondern zeigt auch die Methode auf, der sich die späteren Forschungen bedienen; zur Entstehung und zu den Intentionen dieses bahnbrechenden Werkes sei auf [22] verwiesen. In zwei Zitaten soll Hilberts Standpunkt deutlich gemacht werden:

Wenn ich unter meinen Punkten irgendwelche Systeme von Dingen, z. B. das System Liebe, Gesetz, Schornsteinfeger denke und dann nur meine sämtlichen Axiome als Beziehungen zwischen diesen Dingenannehme, so gelten meine Sätze, z. B. der Pythagoras, auch von diesen Dingen. [7]

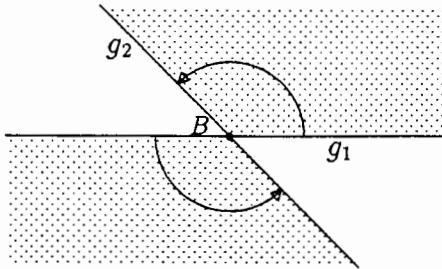
Man muß jederzeit an Stelle von „Punkten“, „Geraden“, „Ebenen“, „Tische“, „Stühle“, „Bierseidel“ sagen können. [3, Seiten 402–403]

Nachstehend analysieren wir in diesem Sinne einige klassische Sätze aus der Geometrie der Kreise und Winkel, insbesondere die Sätze von A. Miquel¹ (Satz 10), J.-V. Poncelet (1788–1867, Satz 11) und W. Wallace (1768–1843, Satz 13).

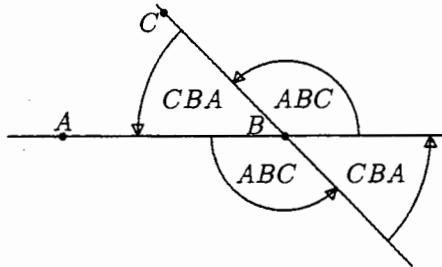
Es sei \mathcal{P} eine Menge. Dabei soll es sich um die Punktmenge der von uns behandelten ebenen Geometrie handeln und deshalb sprechen wir die Elemente von \mathcal{P} als *Punkte* an. Eine 3-Menge von Punkten, das heißt, eine Menge aus drei paarweise verschiedenen Elementen von \mathcal{P} nennen wir *Dreieck*, eine 4-Menge *Viereck*. Ein Dreieck $\{A, B, C\}$ – A , B und C sind seine *Ecken* – wird abgekürzt durch $[ABC]$ (also auch durch $[CBA], \dots$) bezeichnet. Entsprechend bezeichnet $[ABCD]$ das Viereck $\{A, B, C, D\}$ mit den *Ecken* A, B, C, D . Man beachte, daß Begriffe wie „Gerade“ oder „kollinear“ noch nicht zur Verfügung stehen. Das bedeutet, daß bei einer Interpretation in der Anschauungsebene die drei Ecken eines Dreiecks oder die vier Ecken eines Vierecks durchaus in einer Geraden liegen können; sie müssen nur paarweise verschieden sein. Ebenso ist möglich, daß bei einem Viereck drei Ecken in einer Geraden liegen, die vierte aber nicht zu der Geraden gehört. Wir werden später allerdings die Dreiecke mit allen Ecken in einer Geraden als „entartet“ kennzeichnen.

Anschaulich stellen wir uns unter einem „Winkel“ ein Paar von Geraden (g_1, g_2) zusammen mit einem gemeinsamen Punkt B vor; das zugehörige „Winkelfeld“ ist das Gebiet, das die erste der beiden Geraden, g_1 , überstreicht, wenn man sie im mathematisch positiven Sinn (= gegen den Uhrzeigersinn) um den Punkt B dreht, bis sie mit der zweiten Geraden, g_2 zur Deckung kommt. Dabei wird ein Winkel mit seinem Scheitelwinkel vereinigt; die Unterscheidung zwischen beiden entfällt also.

¹Lebensdaten von Auguste Miquel sind mir zur Zeit nicht bekannt; einen Anhaltspunkt gibt nur das Erscheinungsdatum seiner vielzitierten Arbeit [15]. Aus anderen Arbeiten geht hervor, daß er Mathematiklehrer in Nantua (französische Alpen) [14] und Castres (Region Midi-Pyrénées) [16] gewesen ist



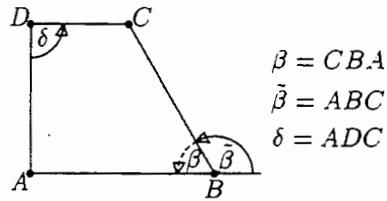
Diesen Zugang hat unseres Wissens erstmalig der amerikanische Mathematiker Roger Arthur Johnson (1889–1954, Professor am Hunter College und am Brooklyn College, beide in New York) gewählt und mit Erfolg beschritten (siehe [8, 9, 10, 11]). Die Nützlichkeit dieser Betrachtungsweise hat auch Eberhard M. Schröder herausgestellt (siehe vor allem [20], außerdem noch [21]); man vergleiche auch die einschlägigen Überlegungen von Schaeffer [19] und Alpers [1]. Für unsere axiomatische Betrachtung wählen wir jedoch einen etwas anderen Ausgangspunkt, behalten aber den Johnsonschen Ansatz als Motivation; in diesem Sinne ist auch das später genannte euklidische Modell zu verstehen. Wir definieren einen *Winkel* als Punktetripel (A, B, C) , kurz ABC , mit $A \neq B \neq C$; der Punkt B heißt *Scheitel* des Winkels ABC . Ist $A = C$, so sprechen wir von einem *trivialen* Winkel. Man beachte, daß die Winkel ABC und CBA im Falle $A \neq C$ zwar den gleichen Scheitel haben, aber trotzdem verschieden sind; wir sprechen von einem *Paar von Nebenwinkel*n. Zu jedem Winkel gibt es genau einen Nebenwinkel.



Zu einem Dreieck $[ABC]$ gehören sechs nichttriviale Winkel ABC , CBA , BAC , ..., die sich in drei Paare von Nebenwinkel ordnen lassen. Ist ABC ein nichttrivialer Winkel, so heißt das aus den gleichen Punkten gebildete Dreieck $[ABC]$ *unterliegendes* Dreieck von ABC .

Ein Viereck enthält vier Dreiecke und liefert daher $24 (= 4 \cdot 6)$ nichttriviale Winkel. Zwei Winkel eines Vierecks heißen *gegenüberliegend*, wenn sie sich

nur im Scheitel unterscheiden. Diese Begriffsbestimmung ist formal naheliegend und für unsere folgenden Entwicklungen auch sehr praktisch, obwohl sie der Anschauung vielleicht etwas widerspricht. Dem Winkel ADC im Vierereck $[ABCD]$ beispielshalber liegt der Winkel ABC gegenüber, und nicht, wie man vielleicht erwarten würde, der Winkel CBA (siehe nachfolgende Figur).



Zu jedem Winkel eines Vierecks gibt es genau einen gegenüberliegenden Winkel. Damit ordnen sich alle Winkel eines Vierecks 12 Paaren von gegenüberliegenden Winkeln.

Bis jetzt haben wir nur eine Menge \mathcal{P} ohne jede zusätzliche Struktur. Eine solche erhalten wir durch Vorgabe einer Äquivalenzrelation *Kongruenz* („ \equiv “) auf der Menge aller Winkel; daß diese zwischen ABC und $\tilde{A}\tilde{B}\tilde{C}$ besteht, schreiben wir als

$$ABC \equiv \tilde{A}\tilde{B}\tilde{C}.$$

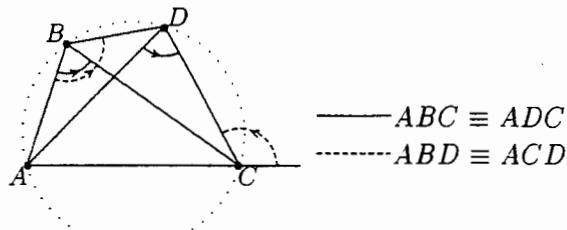
Die Kongruenz soll verschiedenen Bedingungen, sogenannten *Axiomen*, genügen. Wenn immer man in dieser Weise ein Axiomensystem aufbaut, stellt sich die Frage nach der Widerspruchsfreiheit und der Unabhängigkeit. Das im folgenden vorgestellte System ist widerspruchsfrei, jedenfalls in dem gleichen Maße, wie es die euklidische Ebene ist; denn es besitzt in der euklidischen Ebene ein Modell, indem man nämlich $ABC \equiv \tilde{A}\tilde{B}\tilde{C}$ als Existenz einer orientationserhaltenden Kongruenzabbildung interpretiert, die (AB, BC) in $(\tilde{A}\tilde{B}, \tilde{B}\tilde{C})$ überführt. Die Frage nach der Unabhängigkeit untersuchen wir hier nicht. Wir beginnen mit den folgenden Axiomen:

(N) (Nebenwinkelaxiom) Aus der Kongruenz zweier Winkel folgt die Kongruenz ihrer Nebenwinkel:

$$ABC \equiv \tilde{A}\tilde{B}\tilde{C} \Rightarrow CBA \equiv \tilde{C}\tilde{B}\tilde{A}.$$

(G) (Gegenwinkelaxiom) Sind in einem Viereck $[ABCD]$ die einander gegenüberliegenden Winkel ABC und ADC kongruent, so sind es auch

die einander gegenüberliegenden Winkel ABD und ACD .



Schon diese beiden Bedingungen allein ermöglichen den Beweis des wichtigsten Satzes in der Theorie der ebenen Winkel:

Satz 1 (Umfangswinkelsatz) *Ist in einem Viereck ein Paar von gegenüberliegenden Winkeln zueinander kongruent, so gilt dies für jedes Paar von gegenüberliegenden Winkeln.*

Beweis. Es sei ein Viereck $[ABCD]$ mit $ABC \equiv ADC$ gegeben. Aus (G) folgt zunächst $ABD \equiv ACD$. Die Symmetrie der Kongruenzrelation liefert zusammen mit (G) auch gleich $ADB \equiv ACB$. Mit Hilfe von (N) erhalten wir für die entsprechenden Nebenwinkel: $CBA \equiv CDA$, $DBA \equiv DCA$, $BDA \equiv BCA$. Wiederholte Anwendung von (G) ergibt $CBD \equiv CAD$, $DBC \equiv DAC$, $BDC \equiv BAC$. Noch einmal Symmetrie und (G) angewandt, liefert die Kongruenz der noch fehlenden drei Paare von gegenüberliegenden Winkeln: $CAB \equiv CDB$, $DAB \equiv DCB$, $BAD \equiv BCD$. \square

Zu einem nichttrivialen Winkel ABC bilden wir die Punktmenge

$$k(ABC) = \{A, C\} \cup \{V \in \mathcal{P} : A \neq V \neq C \wedge AVC \equiv ABC\}.$$

Lemma 2 *Die Punktmenge $k(ABC)$ ist invariant unter Permutationen der Punkte A , B , C .*

Beweis. Da die Gruppe der Permutationen von den Transpositionen (AB) und (AC) erzeugt wird, genügt es die Invarianz für diese beiden speziellen Permutationen nachzuweisen. Aus (N) folgt unmittelbar $k(ABC) = k(CBA)$. Also bleibt nur noch $k(ABC) = k(BAC)$, das heißt,

$$V \in k(ABC) \iff V \in k(BAC) \quad \text{für alle } V \in \mathcal{P},$$

zu begründen. Dies gilt nach Definition der Punktmengen $k(-)$ unmittelbar für die Punkte $V \in \{A, B, C\}$. Es genügt einen Punkt $V \notin \{A, B, C\}$ zu betrachten; für ihn gilt: $V \in k(ABC) \Leftrightarrow AVC \equiv ABC \Leftrightarrow BVC \equiv BAC \Leftrightarrow V \in k(BAC)$ nach dem Umfangswinkelsatz. \square

Die Menge $k(ABC)$ hängt also nur vom dem unterliegenden Dreieck $[ABC]$, aber nicht von der speziellen Wahl eines von den drei Punkten A, B und C gebildeten Winkels ab; wir nennen sie *Umkreis* des Dreiecks $[ABC]$. Allgemein bezeichnen wir eine Punktmenge als *Kreis*, wenn sie Umkreis eines Dreiecks ist. Wir nennen weiterhin Punkte A, B, C, D (nicht notwendig paarweise verschieden) *konzyklisch*, wenn sie Elemente eines Kreises sind.

Satz 3 (Austauschsatz) $D \in k(ABC) \setminus \{A, C\} \Rightarrow k(ABC) = k(ADC)$.

Beweis. Nach Voraussetzung gilt $ADC \equiv ABC$, woraus $B \in k(ADC)$ folgt. Damit genügt es einen beliebigen Punkt $V \notin \{A, B, C, D\}$ zu betrachten. Wir schließen: $V \in k(ABC) \Leftrightarrow AVC \equiv ABC \Leftrightarrow AVC \equiv ADC \Leftrightarrow V \in k(ADC)$. \square

Folgerung 4 *Jedes Dreieck liegt in genau einem Kreis.*

Beweis. Es ist nur die Eindeutigkeit zu zeigen. Dazu genügt es, folgende Behauptung nachzuweisen: Ist das Dreieck $[ABC]$ in dem Kreis $k(\tilde{A}\tilde{B}\tilde{C})$ enthalten, so gilt $k(\tilde{A}\tilde{B}\tilde{C}) = k(ABC)$. Sei also $[ABC] \subseteq k(\tilde{A}\tilde{B}\tilde{C})$ gegeben. Die 3-Mengen $\{A, B, C\}$ und $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ brauchen nicht disjunkt zu sein. Da es aber auf die Reihenfolge der Punkte nicht ankommt, kann man durch Umbenennung erreichen, daß im Falle von genau einem gemeinsamen Element $A = \tilde{A}$, im Falle von genau zwei gemeinsamen Elementen $A = \tilde{A}$ und $C = \tilde{C}$, und im Falle von drei Elementen $A = \tilde{A}$, $B = \tilde{B}$ und $C = \tilde{C}$ gilt. Im dritten Fall ist nichts mehr zu beweisen, und im zweiten Fall folgt die Behauptung sofort aus dem Austauschsatz. Im ersten Fall (ein gemeinsames Element) folgt aus dem Austauschsatz zunächst $k(\tilde{A}\tilde{B}\tilde{C}) = k(A\tilde{B}\tilde{C}) = k(AB\tilde{C})$ und damit sind wir wieder im Fall von zwei gemeinsamen Elementen. Sind die 3-Mengen $\{A, B, C\}$ und $\{\tilde{A}\tilde{B}\tilde{C}\}$ disjunkt, so liefert der Austauschsatz $k(\tilde{A}\tilde{B}\tilde{C}) = k(\tilde{A}B\tilde{C})$, und damit sind wir wieder bei dem Fall mit einem gemeinsamen Element angelangt. \square

Als nächstes müssen wir einige spezielle Dreiecksklassen kennzeichnen. Ein Dreieck $[ABC]$ heißt *entartet*, wenn alle seine sechs Winkel zu derselben Kongruenzklasse gehören, andernfalls *nichtentartet*.

Lemma 5 Ein Dreieck $[ABC]$ ist genau dann entartet, wenn die beiden folgenden Bedingungen erfüllt sind:

1. Einer seiner Winkel ist kongruent zu den beiden sich durch zyklische Vertauschungen ergebenden Winkeln, zum Beispiel:

$$ABC \equiv BCA \equiv CAB,$$

2. Einer seiner Winkel ist kongruent zu seinem Nebenwinkel, zum Beispiel

$$BAC \equiv CAB.$$

Beweis. Es ist nur zu zeigen, daß die angegebenen Bedingungen hinreichend sind. Durch Umnumerieren läßt sich immer die als Beispiel genommene Situation erreichen. Dann ist noch zu zeigen, daß auch die Winkel CBA und ACB zu den anderen vier Winkeln kongruent sind. Aus (N) ergibt sich

$$ABC \equiv CAB \implies CBA \equiv BAC,$$

$$BCA \equiv CAB \implies ACB \equiv BAC.$$

□

Das legt es nahe, die in dem Lemma genannten Bedingungen einzeln zu betrachten: Ein nichtentartetes Dreieck $[ABC]$ heißt

1. *gleichschenklig*, wenn einer seiner Winkel kongruent zu einem durch zyklische Vertauschungen entstehenden ist.
2. *gleichseitig*, wenn einer seiner Winkel kongruent zu den beiden durch zyklische Vertauschungen entstehenden Winkeln ist.
3. *rechtwinklig*, wenn einer seiner Winkel kongruent zu seinem Nebenwinkel ist.

Ein Winkel ABC heißt ein *rechter Winkel*, wenn er kongruent zu seinem Nebenwinkel und das unterliegende Dreieck nicht entartet ist.

In der bisherigen Allgemeinheit lassen sich die in der Einleitung genannten Aussagen noch nicht beweisen. Wir stellen nun einige weitere Bedingungen an die Kongruenzrelation. Dabei bezeichnen wir einen Winkel als *Nullwinkel*, wenn er zu einem trivialen Winkel kongruent ist; wir bemerken, daß nach (N) der Nebenwinkel eines Nullwinkels ebenfalls ein Nullwinkel ist.

- (V) (*Vertauschungsaxiom*) Aus $ABC \equiv \tilde{A}\tilde{B}\tilde{C}$ folgt $A\tilde{B}\tilde{A} \equiv C\tilde{B}\tilde{C}$. Dabei handelt es sich um einen Spezialfall des Schenkelaustauschsatzes in [13, Seite 6].
- (T1) (*1. Trivialitätsaxiom*) Je zwei triviale Winkel mit verschiedenen Scheiteln sind kongruent.
- (T2) (*2. Trivialitätsaxiom*) Ist ABC ein nichttrivialer Nullwinkel, so ist auch BCA ein Nullwinkel.
- (A) (*Außenwinkelaxiom*) In einem Dreieck $[ABC]$ mit $ABC \equiv BAC$ ist BCA ein Nullwinkel.

Die Benennung des Axioms (A) bedarf einer kurzen Erläuterung: In der klassischen euklidischen Geometrie gilt der Außenwinkelsatz *Ein Außenwinkel eines Dreiecks ist gleich der Summe der nichtanliegenden Innenwinkel*. Ist nun – wie in (A) vorausgesetzt – ein Außenwinkel gleich einem nichtanliegenden Innenwinkel, so muß der andere nichtanliegende Innenwinkel ein Nullwinkel sein.

Die grundlegende Bedeutung dieser Axiome ergibt sich aus folgenden Aussagen.

Lemma 6 *Je zwei triviale Winkel sind kongruent.*

Beweis. Wir betrachten die trivialen Winkel ABA und $\tilde{A}\tilde{B}\tilde{A}$. Ist $B \neq \tilde{B}$, so liefert (T1) unmittelbar die Behauptung. Ist $B = \tilde{B}$, so folgt die Behauptung aus (V) und $A\tilde{B}\tilde{A} \equiv A\tilde{B}\tilde{A}$. \square

Folgerung 7 *Je zwei Nullwinkel sind kongruent.* \square

Lemma 8 *Ein Dreieck $[ABC]$ ist genau dann entartet, wenn ABC ein Nullwinkel ist.*

Beweis. „ \Rightarrow “: Ist $[ABC]$ entartet, so sind alle auftretenden Winkel paarweise kongruent. Insbesondere ist $CAB \equiv ACB$; also ist ABC nach (A) ein Nullwinkel.

„ \Leftarrow “: Da $[ABC]$ ein Dreieck ist, ist ABC ein nichttrivialer Nullwinkel und so ist nach (T2) auch BCA ein nichttrivialer Nullwinkel. (T2) noch einmal

angewandt, erweist auch CAB als Nullwinkel. Dann sind weiterhin die Nebenwinkel der genannten Winkel Nullwinkel. Also sind alle Winkel des Dreiecks $[ABC]$ Nullwinkel und damit paarweise kongruent (Folgerung 7), das heißt, $[ABC]$ ist ein entartetes Dreieck. \square

Satz 9 Es sei $[ABC]$ ein Dreieck.

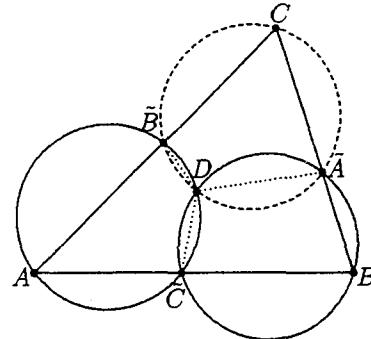
1. Gibt es einen Punkt $D \neq B$ mit $ABD \equiv CBD$, so ist $[ABC]$ entartet.
2. Ist $[ABC]$ entartet, so gilt $ABD \equiv CBD$ für alle Punkte $D \neq B$.

Beweis. 1. Aus $ABD \equiv CBD$ folgt mit (V) $ABC \equiv DBD$, das heißt, ABC ist ein Nullwinkel. Damit ergibt sich die Behauptung aus Lemma 8.

2. Ist $[ABC]$ entartet, so ist ABC nach Lemma 8 ein Nullwinkel, also nach Folgerung 7 $ABC \equiv DBD$ für alle Punkte $D \neq B$. (V) liefert dann $ABD \equiv CBD$ ebenso für alle Punkte $D \neq B$. \square

Wir kommen nun zum ersten der eingangs angekündigten Sätze.

Satz 10 (Miquel 1838 [15]) Es seien ein nicht entartetes Dreieck $[ABC]$ und Punkte $\tilde{A}, \tilde{B}, \tilde{C}$ so gegeben, daß $[\tilde{A}BC], [\tilde{A}\tilde{B}C], [\tilde{A}\tilde{B}\tilde{C}]$ entartete Dreiecke sind. Haben die Kreise $k(\tilde{A}\tilde{B}\tilde{C})$ und $k(\tilde{A}BC)$ noch einen von \tilde{A}, \tilde{B} und \tilde{C} verschiedenen Punkt D gemeinsam, so gehört dieser Punkt D auch zu dem Kreis $k(\tilde{A}BC)$.



Beweis. Aus der Konzyklizität von A, \tilde{B}, \tilde{C} und D folgt nach dem Umfangswinkelsatz, beziehungsweise im Fall $D = A$ nach Lemma 6:

$$(1) \quad D\tilde{C}A \equiv D\tilde{B}A.$$

Aus der Konzyklizität von \tilde{A} , B , \tilde{C} und D folgt nach dem Umfangswinkelsatz, beziehungsweise im Fall $D = B$ nach Lemma 6:

$$(2) \quad D\tilde{A}B \equiv D\tilde{C}B.$$

Da das Dreieck $[AB\tilde{C}]$ entartet ist haben wir nach (N) und Satz 9.2:

$$(3) \quad D\tilde{C}A \equiv D\tilde{C}B.$$

Aus (1), (2) und (3) folgt nun:

$$(4) \quad D\tilde{A}B \equiv D\tilde{B}A.$$

Da die Dreiecke $[\tilde{A}BC]$, $[A\tilde{B}C]$ entartet sind, haben wir, wiederum nach (N) und 9.2:

$$(5) \quad D\tilde{A}B \equiv D\tilde{A}C$$

$$(6) \quad D\tilde{B}A \equiv D\tilde{B}C.$$

Aus (4), (5) und (6) folgt nun:

$$(7) \quad D\tilde{A}C \equiv D\tilde{B}C;$$

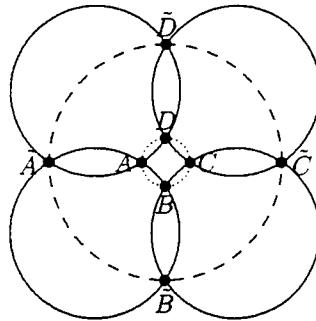
das bedeutet die behauptete Konzyklizität der Punkte \tilde{A} , \tilde{B} , C und D . \square

Bemerkungen Beim Satz von Miquel handelt es sich um eine Aussage, die im Zusammenhang mit den Grundlagen der Geometrie als „Schließungssatz“ bezeichnet wird. Unsere Formulierung lehnt sich an die Fassung an, die Johnson in seinem weit verbreiteten Lehrbuch [11, Nr. 184] angibt. Mit den bisher entwickelten Hilfsmitteln läßt sich jedoch der folgende Ausartungsfall des Satzes von Miquel noch nicht behandeln:

Es seien ein nicht entartetes Dreieck $[ABC]$ und Punkte \tilde{A} , \tilde{B} , \tilde{C} so gegeben, daß $[\tilde{A}BC]$, $[A\tilde{B}C]$, $[AB\tilde{C}]$ entartete Dreiecke sind. Haben die Kreise $k(A\tilde{B}\tilde{C})$ und $k(\tilde{A}BC)$ nur den Punkt \tilde{C} gemeinsam, so gehört \tilde{C} auch zu dem Kreis $k(\tilde{A}BC)$.

Klassisch wird hierzu der Satz vom Sehnen-Tangenten-Winkel benötigt, den wir uns für eine weitere Arbeit vorbehalten.

B. L. van der Waerden (* 1903) und L. J. Smid [23] haben herausgearbeitet, daß im Zusammenhang mit Grundlagenfragen eine andere, 1845 angegebene Version des Satzes von Miquel [16, Seite 23] von großer Bedeutung ist:



(man betrachte dazu die Figur:) Es seien vier Kreise (durchgehogene Linien) gegeben, die sich in den Punkten $A, \tilde{A}, B, \tilde{B}, C, \tilde{C}, D, \tilde{D}$ schneiden. Liegen die Punkte A, B, C, D auf einem Kreis (punktige Linie), so liegen auch die Punkte $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ auf einem Kreis gestrichelte Linie.

Auch diese Behauptung lässt sich mit unseren Methoden noch nicht beweisen. Klassisch erhält man sie aus der ursprünglichen Formulierung mit Hilfe der sogenannten *Kreisspiegelung* [25, Seiten 178–180]. Eine Übersicht über weitere Versionen des Satzes von Miquel enthält [13, Figur 4].

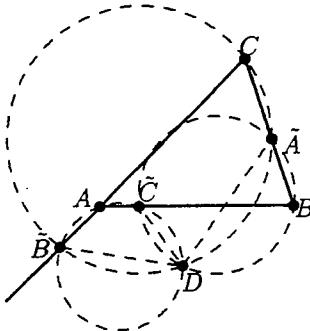
Der nachfolgende Satz steht in engem Zusammenhang mit dem Satz von Miquel, wie die zum Beweis angegebene Figur zeigt.

Satz 11 (Poncelet 1822 [18, § 468]) Es seien ein nicht entartetes Dreieck $[ABC]$, ein Dreieck $[\tilde{A}\tilde{B}\tilde{C}]$ und ein Punkt $D \notin \{A, B, C, \tilde{A}, \tilde{B}, \tilde{C}\}$ so gegeben, daß $[\tilde{A}\tilde{B}\tilde{C}]$, $[\tilde{A}B\tilde{C}]$, $[AB\tilde{C}]$ entartete Dreiecke sind und

$$(1) \quad A\tilde{B}D \equiv B\tilde{C}D \equiv C\tilde{A}D,$$

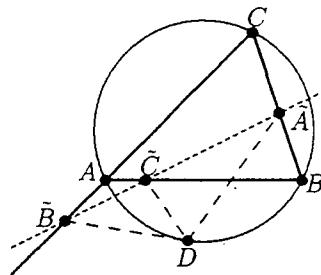
ist. Dann gilt:

$$(2) \quad D \in k(A\tilde{B}\tilde{C}) \cap k(\tilde{A}B\tilde{C}) \cap k(\tilde{A}\tilde{B}C).$$



und

$$(3) \quad D \in k(ABC) \Leftrightarrow [\tilde{A}\tilde{B}\tilde{C}] \text{ entartet.}$$



Beweis. Da $[\tilde{A}\tilde{B}\tilde{C}]$ entartet ist, ist nach Satz 9.2 und Voraussetzung

$$C\tilde{B}D \equiv A\tilde{B}D \equiv C\tilde{A}D,$$

woraus die Konzyklizität der Punkte $\tilde{A}, \tilde{B}, C, D$ folgt. Analog ergeben sich die Konzyklizität der Punkte $\tilde{A}, A, \tilde{C}, D$ und die Konzyklizität der Punkte $A, \tilde{B}, \tilde{C}, D$. Also haben wir (2).

Wir berechnen nun:

$$\begin{aligned} (4) \quad ABC &\equiv \tilde{C}BC && \text{nach Satz 9.2, da } [\tilde{A}\tilde{B}\tilde{C}] \text{ entartet,} \\ &\equiv \tilde{C}B\tilde{A} && \text{nach (N) und Satz 9.2, da } [\tilde{A}\tilde{B}\tilde{C}] \text{ entartet,} \\ &\equiv \tilde{C}D\tilde{A} && \text{nach (2).} \end{aligned}$$

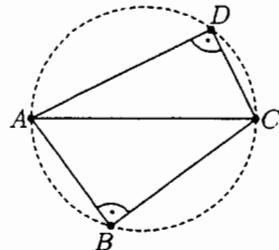
Damit schließen wir:

$$\begin{aligned} D \in k(ABC) &\Leftrightarrow ADC \equiv ABC && \text{Definition des Umkreises} \\ &\Leftrightarrow ADC \equiv \tilde{C}D\tilde{A} && \text{wegen (4)} \\ &\Leftrightarrow AD\tilde{C} \equiv CD\tilde{A} && \text{Vertauschungsaxiom (V)} \\ &\Leftrightarrow A\tilde{B}\tilde{C} \equiv CD\tilde{A} && \text{nach (2)} \\ &\Leftrightarrow A\tilde{B}\tilde{C} \equiv C\tilde{B}\tilde{A} && \text{nach (2)} \\ &\Leftrightarrow A\tilde{B}\tilde{C} \equiv A\tilde{B}\tilde{A} && [\tilde{A}\tilde{B}\tilde{C}] \text{ entartet und Satz 9.2} \\ &\Leftrightarrow [\tilde{A}\tilde{B}\tilde{C}] \text{ entartet} && (\text{N}) \text{ und Satz 9.1} \end{aligned}$$

□

Zum Beweis des Satzes von Wallace (Satz 13) benötigen wir nun noch ein Axiom.

- (T) Sind ABC und ADC rechte Winkel, so sind die Punkte A, B, C und D konzyklisch.



Das Symbol „T“ ehrt natürlich Thales von Milet; das Axiom ist ja ein Teil des berühmten Satzes vom Thaleskreis. Der noch fehlende Teil dieses Satzes lässt sich nun beweisen.

Satz 12 (Thales ≈ 600 v. Chr.) Es sei ABC ein rechter Winkel. Dann gilt für einen Punkt $D \in \mathcal{P} \setminus \{A, C\}$:

$$D \in k(ABC) \Leftrightarrow ADC \text{ rechter Winkel.}$$

Beweis. „ \Rightarrow “: Aus $D \in k(ABC)$ folgt nach Definition des Umkreises

$$(1) \quad ADC \equiv ABC.$$

Da nach Voraussetzung ABC ein rechter Winkel ist, ist $[ABC]$ nicht entartet. Also ist nach Lemma 8 ABC kein Nullwinkel. Wegen (1) ist dann auch ADC kein Nullwinkel und damit ist – wiederum nach Lemma 8 – das Dreieck ADC nicht entartet. Es bleibt

$$(2) \quad ADC \equiv CDA$$

zu zeigen. Weil ABC ein rechter Winkel ist, haben wir auch noch:

$$(3) \quad ABC \equiv CBA.$$

Aus (1) und (3) folgt:

$$(4) \quad ADC \equiv CBA.$$

Das Nebenwinkelaxiom (N) liefert nun

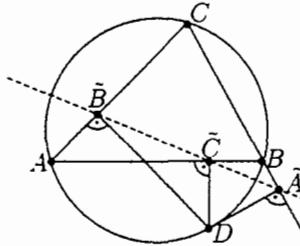
$$(5) \quad CDA \equiv ABC.$$

Die Behauptung (2) ergibt sich nun aus (1) und (5).

„ \Leftarrow “: Axiom (T). □

Satz 13 (Wallace 1797 [24]) Es seien ein nicht entartetes Dreieck $[ABC]$, ein Dreieck $[\tilde{A}\tilde{B}\tilde{C}]$ und ein Punkt $D \notin \{A, B, C, \tilde{A}, \tilde{B}, \tilde{C}\}$ so gegeben, daß $[\tilde{A}\tilde{B}\tilde{C}]$, $[A\tilde{B}C]$, $[AB\tilde{C}]$ entartete Dreiecke und $A\tilde{B}D$, $B\tilde{C}D$, $C\tilde{A}D$ rechte Winkel sind. Dann gilt:

$$D \in k(ABC) \Leftrightarrow [\tilde{A}\tilde{B}\tilde{C}] \text{ entartet.}$$



Beweis. Da das Dreieck $[AB\tilde{C}]$ entartet ist, folgt aus Satz 9.2

$$(1) \quad B\tilde{C}D \equiv A\tilde{C}D;$$

wegen (N) ist mit $B\tilde{C}D$ auch $A\tilde{C}D$ ein rechter Winkel. Aus (T) folgt nun die Konzyklizität der Punkte $A, \tilde{B}, \tilde{C}, D$ und der Umfangswinkelsatz liefert

$$(2) \quad A\tilde{B}D \equiv A\tilde{C}D.$$

Aus (1) und (2) folgt

$$A\tilde{B}D \equiv B\tilde{C}D.$$

Analog ergibt sich:

$$B\tilde{C}D \equiv C\tilde{A}D.$$

Damit ist die Voraussetzung (1) des Satzes von Poncelet (Satz 11) erfüllt, und dieser liefert nun unmittelbar den Satz von Wallace. \square

Bemerkungen. Zum Vergleich geben wir hier noch die klassische Formulierung des Satzes von Wallace an:

Fällt man von einem Punkt D der Ebene die Lote auf die Seiten eines Dreiecks $[ABC]$, so liegen die drei Lotfußpunkte genau dann auf einer Geraden, wenn D auf dem Umkreis von $[ABC]$ liegt.

Ein Beweis dieses Satzes mit den klassischen synthetischen Methoden findet sich in [4, Satz 2.51, Seiten 45–46]. Ähnlich unserem Vorgehen ist die Darstellung in [17, Seiten 255–257]; die hier gegebene Begründung ist insofern etwas einfacher, als von einer „Winkeladdition“ kein expliziter Gebrauch gemacht wird. Wir müssen allerdings einige Grenzfälle von der Betrachtung ausschließen; sie sind in [17] ausführlich behandelt.

Dasselbe gilt für Herstellung einer Verbindung zu den affinen Inzidenzebenen, oder besser zu den miquelschen Möbiusebenen (siehe [2]), und für den Ausbau zu einem Axiomensystem, das die klassische euklidische Ebene charakterisiert.

Für zahlreiche Hinweise danke ich meinen Kollegen Pickert (Gießen), Schröder (Hamburg) und Seebach (München).

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Epimorphisms of Separated Superconvex Spaces

Ralf Kemper

Abstract: We give characterizations of the epimorphisms in the categories of (pre)separated convex and separated superconvex spaces.

AMS Subject Classification (1991): 18A20, 18C20.

Key words: Epimorphism, (separated) superconvex space, (pre)separated convex space.

Introduction

In [9] Pumplün and Röhrl introduced the category \mathbf{TC} (resp. \mathbf{TC}_{fin}) of (finitely) totally convex (t.c.) spaces, which are the Eilenberg-Moore algebras of the monad induced by the unit ball functor from the category of Banach spaces (resp. normed vector spaces) with linear contractions to the category of sets. We refer to [9] and [10] for all definitions and conventions, but in accordance with [3] we use the term “absolutely convex” for the spaces Pumplün and Röhrl call “finitely totally convex”; \mathbf{AC} denotes the category of absolutely convex spaces.

Later Pumplün resp. Pumplün and Röhrl introduced the categories \mathbf{PC} of positively convex spaces [8], \mathbf{SC} of superconvex spaces [7] resp. \mathbf{Conv} of convex spaces [11]. In [12] they characterize the epimorphisms in the category of separated (finitely) t.c. spaces. In [5] the author gave two other characterizations of the epimorphisms in these categories, and in [3], [4] characterizations of the epimorphisms in the categories of (absolutely) convex resp. discrete t.c. (introduced in [6]) spaces.

The category \mathbf{SC}_{sep} of separated superconvex spaces is defined as the co-generator hull of the unit intervall $[0, 1]$ in \mathbf{SC} ([2], 3.1). In [2], 3.6, one finds a charaterization of separated superconvex spaces, i.e. a superconvex space is separated if and only if it is preseparated, which means, that a well-known cancellation law ([11], 1.4) is fulfilled.

In this paper a characterization of the epimorphisms in the categories of separated (super)convex and preseparated convex spaces is given. \square

§1 Basic Properties

A totally convex (t.c.) space D is a non-empty set which admits every $\alpha \in \Omega := \{(\alpha_i)_{i \in \mathbb{N}} \in K^{\mathbb{N}} \mid \sum_i |\alpha_i| \leq 1\}$, $K \in \{\mathbb{R}, \mathbb{C}\}$, as \mathbb{N} -ary operation. The result of such an operation is written as formal sum $\sum_i \alpha_i x_i$, ($x_i \in D$ ($i \in \mathbb{N}$)), $\alpha \in \Omega$), and the operations are required to satisfy the following two axioms.

$$(\text{TC1}) \quad \sum_i \delta_i^j y_i = y_j, \text{ for all } j \in \mathbb{N}, \quad (y_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}.$$

$$(\text{TC2}) \quad \sum_i \alpha_i (\sum_j \beta_j^i y_j) = \sum_j (\sum_i \alpha_i \beta_j^i) y_j, \text{ for all } \alpha, \beta^i \in \Omega, y_i \in X \quad (i \in \mathbb{N}).$$

A morphism $f : C \longrightarrow D$ between t.c. spaces is a set mapping preserving these operations, i.e. $f(\sum_i \alpha_i x_i) = \sum_i \alpha_i f(x_i)$, $x_i \in C$ ($i \in \mathbb{N}$), $\alpha \in \Omega$ ([9], 2.2).

We define $\Omega_{fin} := \{\alpha \in \Omega \mid |\text{supp } \alpha| < \infty\}$, where $\text{supp } \alpha := \{i \in \mathbb{N} \mid \alpha_i \neq 0\}$ for $\alpha \in \Omega$. An absolutely convex space (a.c.) is a non-empty Ω_{fin} -algebra satisfying (TC1) and the restriction of (TC2) to $\alpha, \beta^i \in \Omega_{fin}$ ($i \in \mathbb{N}$); \mathbf{AC} denotes the category of a.c. spaces and maps that preserve the Ω_{fin} -operations. Furthermore, in case $K = \mathbb{R}$ we define $\Omega^+ := \{\alpha \in \Omega \mid \forall i \in \mathbb{N} \quad \alpha_i \geq 0\}$, $\Omega_{fin}^+ := \Omega^+ \cap \Omega_{fin}$. Restricting the \mathbf{TC} -operations (and thus the axioms and the conditions for morphisms) to Ω^+ (resp. Ω_{fin}^+) we get the category \mathbf{PC} (resp. \mathbf{PC}_{fin}) of (finitely) positively convex spaces ([8], 2.4). Similarly, putting $\Omega_{sc} := \{\alpha \in \Omega^+ \mid \sum_i \alpha_i = 1\}$ (resp. $\Omega_c := \Omega_{sc} \cap \Omega_{fin}$) and discarding the condition $X \neq \emptyset$ yields the category \mathbf{SC} (resp. \mathbf{Conv}) of superconvex (resp. convex) spaces. For \mathbf{TC} , \mathbf{AC} , \mathbf{PC} and \mathbf{PC}_{fin} , $0 := (0)_{i \in \mathbb{N}}$

is an algebra operation, which is constant on any space, hence 0 yields a nullary operation; we call its value the zero element of the space. For **SC** and **Conv** there is no nullary operation and we have to admit the empty space in order to get a complete (and cocomplete) category. Note that any constant mapping between two objects of **SC** (resp. **Conv**) is a morphism in **SC** (resp. **Conv**). □

1.1 Proposition: Let $D \in \mathbf{Conv}$ (resp. $D \in \mathbf{AC}, D \in \mathbf{PC}_{\text{fin}}$) and let C be a non-empty subspace of D . Define the relation “ \sim ” on D by $x \sim y$ if and only if there exist $c, c' \in C$ and $\alpha \in]0, 1]$ with $\alpha x + (1 - \alpha)c = \alpha y + (1 - \alpha)c'$. Then “ \sim ” is a congruence relation on D .

Proof: Let $D \in \mathbf{Conv}$. Obviously, the relation “ \sim ” is reflexive and symmetric. Next we show, that “ \sim ” is compatible. Let $x_i, y_i \in D$ with $x_i \sim y_i$ ($i \in \mathbb{N}$) and $\alpha \in \Omega_c$. Then there exist $c_i, c'_i \in C$ and $\beta_i \in]0, 1]$ with $\beta_i x_i + (1 - \beta_i)c_i = \beta_i y_i + (1 - \beta_i)c'_i$ ($i \in \mathbb{N}$). Since $\text{supp } \alpha$ is finite, $\beta := \inf\{\beta_i | i \in \text{supp } \alpha\} > 0$ holds. In case $\beta = 1$, i.e. $\beta_i = 1$ ($i \in \text{supp } \alpha$), we have $x_i = y_i$ ($i \in \text{supp } \alpha$) and this implies $\sum_i \alpha_i x_i = \sum_i \alpha_i y_i$ ([9], 2.4(iii)). Otherwise, define

$$\bar{c}_i := \frac{\beta(1 - \beta_i)}{\beta_i(1 - \beta)} c_i + \frac{\beta_i - \beta}{\beta_i(1 - \beta)} c'_i \in C \quad (i \in \text{supp } \alpha).$$

Then for all $i \in \text{supp } \alpha$

$$\begin{aligned} \beta x_i + (1 - \beta) \bar{c}_i &= \frac{\beta}{\beta_i} (\beta_i x_i + (1 - \beta_i)c_i) + \frac{\beta_i - \beta}{\beta_i} c'_i \\ &= \frac{\beta}{\beta_i} (\beta_i y_i + (1 - \beta_i)c'_i) + \frac{\beta_i - \beta}{\beta_i} c'_i = \beta y_i + (1 - \beta)c'_i \end{aligned}$$

follows. This implies

$$\begin{aligned} \beta(\sum_i \alpha_i x_i) + (1 - \beta)(\sum_{i \in \text{supp } \alpha} \alpha_i \bar{c}_i) &= \sum_{i \in \text{supp } \alpha} \alpha_i (\beta x_i + (1 - \beta) \bar{c}_i) \\ &= \sum_{i \in \text{supp } \alpha} \alpha_i (\beta y_i + (1 - \beta)c'_i) = \beta(\sum_i \alpha_i y_i) + (1 - \beta)(\sum_{i \in \text{supp } \alpha} \alpha_i c'_i). \end{aligned}$$

Thus we get $\sum_i \alpha_i x_i \sim \sum_i \alpha_i y_i$. Finally, “ \sim ” is shown to be transitive. Let $x, y, z \in D$ with $x \sim y$ and $y \sim z$. Then there exist $c_i, c'_i \in C$ ($i = 0, 1$) and $\beta, \gamma \in]0, 1]$ with $\beta x + (1 - \beta)c_0 = \beta y + (1 - \beta)c'_0$ and $\gamma y + (1 - \gamma)c'_0 =$

$\gamma z + (1 - \gamma)c'_1$. As in the first part of the proof we may assume $\beta = \gamma$. For $c \in C$ define

$$c_2 := \frac{2(1-\beta)}{2-\beta} \left(\frac{1}{2}c_0 + \frac{1}{2}c_1 \right) + \frac{\beta}{2-\beta}c \in C$$

and

$$c'_2 := \frac{2(1-\beta)}{2-\beta} \left(\frac{1}{2}c'_0 + \frac{1}{2}c'_1 \right) + \frac{\beta}{2-\beta}c \in C.$$

This implies

$$\begin{aligned} \frac{\beta}{2}x + \frac{2-\beta}{2}c_2 &= \frac{\beta}{2}x + (1 - \beta) \left(\frac{1}{2}c_0 + \frac{1}{2}c_1 \right) + \frac{\beta}{2}c \\ &= \frac{1}{2}(\beta x + (1 - \beta)c_0) + \frac{1}{2}((1 - \beta)c_1 + \beta c) = \frac{1}{2}(\beta y + (1 - \beta)c'_0) + \frac{1}{2}((1 - \beta)c_1 + \beta c) \\ &= \frac{1}{2}(\beta y + (1 - \beta)c_1) + \frac{1}{2}((1 - \beta)c'_0 + \beta c) = \frac{1}{2}(\beta z + (1 - \beta)c'_1) + \frac{1}{2}((1 - \beta)c'_0 + \beta c) \\ &= \frac{\beta}{2}z + \frac{2-\beta}{2}c'_2 \text{ and hence } x \sim z. \end{aligned}$$

The proof for $D \in \mathbf{AC}$ (resp. $D \in \mathbf{PC}_{\text{fin}}$) is similar. \square

A convex space D is called preseparated, if the so-called cancellation law holds, i.e., if for any $x, y, z \in D$ and any $\alpha \in]0, 1[$, $\alpha x + (1 - \alpha)z = \alpha y + (1 - \alpha)z$ implies $x = y$ ([11], 1.4, 4.9). $D \in \mathbf{SC}$ is called preseparated, if the underlying convex space is ([2], 3.1(i)).

A convex (resp. superconvex) space D is called separated, if and only if the family of **Conv**-morphisms (resp. **SC**-morphisms) $D \rightarrow [0, 1]$, $([0, 1]$ with its canonical (super)convex structure), is point-separating ([11], 4.12 (resp. [2], 3.1(ii)).

A superconvex space D is separated if and only if it is preseparated ([2], 3.6). A convex space D is separated, if and only if the following property is fulfilled: If, for $u, v, x_n, y_n \in D$

$$(1 - \frac{1}{n})u + \frac{1}{n}x_n = (1 - \frac{1}{n})v + \frac{1}{n}y_n \quad (n \in \mathbb{N})$$

holds, then $u = v$ ([2], 2.3).

SC_{sep} (resp. **Conv_{sep}**, **Conv_{psep}**) denotes the full subcategory of **SC** (resp. **Conv**), which is spanned by all (pre)separated superconvex (resp. separated

convex, preseparated convex) spaces. The categories \mathbf{SC}_{sep} (resp. $\mathbf{Conv}_{\text{sep}}$, $\mathbf{Conv}_{\text{psep}}$) are ext-epi-reflective subcategories of \mathbf{SC} (resp. \mathbf{Conv}) ([1], 16.8 (resp. [11], 4.11, 4.13 (i))). \square

1.2 Remark: $\mathbb{R}, \mathbb{R}^+ := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}^{++} := \mathbb{R}^+ \setminus \{0\}$ are (with the canonical structures) preseparated convex spaces. For all $n \in \mathbb{N}$ $(1 - \frac{1}{n})1 + \frac{1}{n}n = (1 - \frac{1}{n})2 + \frac{1}{n}1$ holds. Consequently, $\mathbb{R}^{++} \subset \mathbb{R}^+ \subset \mathbb{R}$ are not separated convex spaces. \square

§2 Epimorphisms of Separated Superconvex Spaces

Let $D \in \mathbf{SC}$. Define a relation “ \sim ” on D by $x \sim y$ if and only if there exist a $a \in D$ with $\frac{1}{2}x + \frac{1}{2}a = \frac{1}{2}y + \frac{1}{2}a$ ($x, y \in D$).

2.1 Proposition: Let $D \in \mathbf{SC}$. Then the above relation “ \sim ” is a congruence relation on D , and the projection $D \rightarrow D/\sim$ gives the reflection into the subcategory \mathbf{SC}_{sep} .

Proof: Obviously, the relation “ \sim ” is reflexive, symmetric and compatible. Next we show, that “ \sim ” is transitive. For this purpose let $x, y, z \in D$ with $x \sim y$ and $y \sim z$. Thus there exist $a, b \in D$ with $\frac{1}{2}x + \frac{1}{2}a = \frac{1}{2}y + \frac{1}{2}a$ and $\frac{1}{2}y + \frac{1}{2}b = \frac{1}{2}z + \frac{1}{2}b$. This implies

$$\begin{aligned} & \frac{1}{4}x + \frac{3}{4}\left(\frac{1}{2}a + \frac{1}{2}b\right) &= \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{2}a\right) + \frac{1}{2}\left(\frac{1}{4}a + \frac{3}{4}b\right) \\ &= \frac{1}{2}\left(\frac{1}{2}y + \frac{1}{2}a\right) + \frac{1}{2}\left(\frac{1}{4}a + \frac{3}{4}b\right) &= \frac{1}{2}\left(\frac{1}{2}y + \frac{1}{2}b\right) + \frac{1}{2}\left(\frac{1}{4}b + \frac{3}{4}a\right) \\ &= \frac{1}{2}\left(\frac{1}{2}z + \frac{1}{2}b\right) + \frac{1}{2}\left(\frac{1}{4}b + \frac{3}{4}a\right) &= \frac{1}{4}z + \frac{3}{4}\left(\frac{1}{2}a + \frac{1}{2}b\right). \end{aligned}$$

Hence, $\frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}a + \frac{1}{2}b\right) = \frac{1}{2}z + \frac{1}{2}\left(\frac{1}{2}a + \frac{1}{2}b\right)$ ([11], 1.2) and $x \sim z$ holds. Thus “ \sim ” is a congruence relation on D .

Let $x, y, z \in D$ and $\beta \in]0, 1[$ with $\beta x + (1 - \beta)z \sim \beta y + (1 - \beta)z$. This implies the existence of an element $a \in D$ with $\frac{1}{2}(\beta x + (1 - \beta)z) + \frac{1}{2}a = \frac{1}{2}(\beta y + (1 - \beta)z) + \frac{1}{2}a$. Put $z' := \frac{1-\beta}{2-\beta}z + \frac{1}{2-\beta}a$. Then we get $\frac{\beta}{2}x + (1 - \frac{\beta}{2})z' = \frac{1}{2}(\beta x + (1 - \beta)z) + \frac{1}{2}a = \frac{1}{2}(\beta y + (1 - \beta)z) + \frac{1}{2}a = \frac{\beta}{2}y + (1 - \frac{\beta}{2})z'$, thus $\frac{1}{2}x + \frac{1}{2}z' = \frac{1}{2}y + \frac{1}{2}z'$

([11], 1.2) and $x \sim y$. Thus D/\sim is preseparated and, by [2], 3.6, D/\sim is separated. $\pi : D \rightarrow D/\sim$ denotes the canonical projection. For some **SC**-morphism $f : D \rightarrow E$, $E \in \mathbf{SC}_{\text{sep}}$, define a mapping $\hat{f} : D/\sim \rightarrow E$ by $\hat{f}(\pi(x)) := f(x)$ ($x \in D$). For $x, y \in D$ with $\pi(x) = \pi(y)$ there exists a $a \in D$ with $\frac{1}{2}x + \frac{1}{2}a = \frac{1}{2}y + \frac{1}{2}a$. This implies $\frac{1}{2}f(x) + \frac{1}{2}f(a) = f(\frac{1}{2}x + \frac{1}{2}a) = f(\frac{1}{2}y + \frac{1}{2}a) = \frac{1}{2}f(y) + \frac{1}{2}f(a)$, thus $f(x) = f(y)$, since E is preseparated. Consequently, \hat{f} is well-defined. Obviously, \hat{f} is uniquely determined by f and \hat{f} is a **SC_{sep}**-morphism, rendering a left adjoint. \square

2.2 Lemma: The forgetful functor $V : \mathbf{TC}_{\text{sep}} \rightarrow \mathbf{SC}_{\text{sep}}$ has a faithful left adjoint and the unit of this adjunction is injective.

Proof: It is easy to see by the Adjoint Functor Theorem that V has a left adjoint. By [2], 3.5, each separated superconvex space is a superconvex subspace of a separated t.c. space. Therefore the unit of the above adjunction is injective and the left adjoint is faithful. \square

Denote by $\widehat{\bigcirc}_{\text{sep}} : \mathbf{Ban}_1 \rightarrow \mathbf{TC}_{\text{sep}}$ the restriction of the comparisons functor $\widehat{\bigcirc} : \mathbf{Ban}_1 \rightarrow \mathbf{TC}$ ([9], 3.5). $\widehat{\bigcirc}$ has a left adjoint $S : \mathbf{TC} \rightarrow \mathbf{Ban}_1$ ([9], 7.7). Thus the restriction $S_{\text{sep}} : \mathbf{TC}_{\text{sep}} \rightarrow \mathbf{Ban}_1$ of S is a left adjoint of $\widehat{\bigcirc}_{\text{sep}}$ with an injective unit ([9], 7.7, [10], 11.1). This yields the following

2.3 Corollary: The functor $V \circ \widehat{\bigcirc}_{\text{sep}} : \mathbf{Ban}_1 \rightarrow \mathbf{SC}_{\text{sep}}$ (see 2.2) has a faithful left adjoint with an injective unit η . \square

If D is a separated superconvex space, and for $K := \mathbb{IR}$, $\eta_D : D \rightarrow \widehat{\bigcirc}(B)$, $B \in \mathbf{Ban}_1$, is as in 2.3 or [2], 3.5, define $\eta' : D \rightarrow \widehat{\bigcirc}(B)$ by $\eta'(x) := \frac{1}{2}\eta_D(x)$ ($x \in D$). Obviously, η' is an injective **SC**-morphism. Now, consider the subset of $\widehat{\bigcirc}(B)$ $\eta'(D) - \eta'(D) := \{\eta'(x) - \eta'(y) \mid x, y \in D\}$. $\widehat{\bigcirc}(B)$ is a real separated t.c. space ([11], 11.2(iii)) and we have

2.4 Lemma: The set $\eta'(D) - \eta'(D)$ is a real separated t.c. subspace of $\widehat{\bigcirc}(B)$.

Proof: The proof is essentially the same as far 2.2 in [2]. Let $x_i, y_i \in D$ ($i \in \mathbb{N}$), $\alpha \in \Omega$ and put $\alpha_0 := 1 - \sum_{i \in \mathbb{N}} |\alpha_i|$. Define elements $u_i, v_i \in D$ ($i \in \mathbb{N}$) by $u_i := x_i$ (resp. $v_i := y_i$) for all $i \in \mathbb{N}$ with $\alpha_i \geq 0$ and $u_i := y_i$ (resp. $v_i := x_i$) otherwise. Let $u_0 := v_0$ be some element of D and define $u := \sum_{i \in \mathbb{N}_0} |\alpha_i| u_i$, $v := \sum_{i \in \mathbb{N}_0} |\alpha_i| v_i$. From $\frac{1}{2} \sum_{i \in \mathbb{N}} \alpha_i (\eta'(x_i) - \eta'(y_i)) = \frac{1}{2} \sum_{i \in \mathbb{N}} |\alpha_i| (\eta'(u_i) - \eta'(v_i))$
 $= \sum_{i \in \mathbb{N}} |\alpha_i| (\frac{1}{2} \eta'(u_i) - \frac{1}{2} \eta'(v_i)) = \frac{1}{2} \sum_{i \in \mathbb{N}} |\alpha_i| \eta'(u_i) - \frac{1}{2} \sum_{i \in \mathbb{N}} |\alpha_i| \eta'(v_i)$
 $= \frac{1}{2} \sum_{i \in \mathbb{N}_0} |\alpha_i| \eta'(u_i) - \frac{1}{2} \sum_{i \in \mathbb{N}_0} |\alpha_i| \eta'(v_i) = \frac{1}{2} \eta'(\sum_{i \in \mathbb{N}_0} |\alpha_i| u_i) - \frac{1}{2} \eta'(\sum_{i \in \mathbb{N}_0} |\alpha_i| v_i)$
 $= \frac{1}{2}(\eta'(u) - \eta'(v))$ we get $\sum_i \alpha_i (\eta'(x_i) - \eta'(y_i)) = \eta'(u) - \eta'(v) \in \eta'(D) - \eta'(D)$. Thus the subset $\eta'(D) - \eta'(D) \subset \widehat{\bigcirc}(B)$ is closed under real TC-operations and, consequently, $\eta'(D) - \eta'(D)$ is a real t.c. subspace of $\widehat{\bigcirc}(B)$. By [10], 11.5(ii), $\eta'(D) - \eta'(D)$ is separated. \square

2.5 Definition: Let $D \in \mathbf{SC}$ (resp. $D \in \mathbf{Conv}$) and C be a subspace of D . \tilde{C} is the smallest subspace of D with $C \subset \tilde{C}$ having for all $y \in D$ the following property: If, for every $\varepsilon \in]0, 1[$, there exist $c, c' \in C$, $a, a' \in D$, $\alpha \in]0, 1[$ with

$$\frac{\alpha}{2}y + \frac{\alpha\varepsilon}{2}a + (1 - \frac{\alpha(1+\varepsilon)}{2})c = \frac{\alpha\varepsilon}{2}a' + (1 - \frac{\alpha\varepsilon}{2})c',$$

then $y \in \tilde{C}$. \square

2.6 Proposition: Let $f, g : D \rightarrow E$ be SC-morphisms and let E be separated. Then the equalizer $\text{equ}(f, g) := \{x \in D \mid f(x) = g(x)\}$ is a subspace of D and $(\text{equ}(f, g))^\sim = \text{equ}(f, g)$ holds.

Proof: Obviously, $\text{equ}(f, g)$ is a subspace of D . Let $y \in D$ such that for every $\varepsilon \in]0, 1[$ there exist $c, c' \in \text{equ}(f, g)$, $a, a' \in D$, $\alpha \in]0, 1[$ with

$$\frac{\alpha}{2}y + \frac{\alpha\varepsilon}{2}a + (1 - \frac{\alpha(1+\varepsilon)}{2})c = \frac{\alpha\varepsilon}{2}a' + (1 - \frac{\alpha\varepsilon}{2})c'.$$

By 2.3 there exists a real Banach space B and $\eta := \eta_E : E \longrightarrow \widehat{\bigcirc}(B)$ is an injective \mathbf{SC} -morphism. This yields

$$\begin{aligned} & \frac{\alpha}{2}(\eta \circ f)(y) + \frac{\alpha\varepsilon}{2}(\eta \circ f)(a) + (1 - \frac{\alpha(1+\varepsilon)}{2})(\eta \circ f)(c) \\ &= \frac{\alpha\varepsilon}{2}(\eta \circ f)(a') + (1 - \frac{\alpha\varepsilon}{2})(\eta \circ f)(c') \end{aligned}$$

and because of $c, c' \in \text{equ}(f, g)$

$$\begin{aligned} & \frac{\alpha}{2}(\eta \circ g)(y) + \frac{\alpha\varepsilon}{2}(\eta \circ g)(a) + (1 - \frac{\alpha(1+\varepsilon)}{2})(\eta \circ f)(c) \\ &= \frac{\alpha\varepsilon}{2}(\eta \circ g)(a') + (1 - \frac{\alpha\varepsilon}{2})(\eta \circ f)(c'). \end{aligned}$$

An easy computation in B yields

$$\begin{aligned} & \frac{\alpha}{2}(\eta \circ f)(y) + \frac{\alpha\varepsilon}{2}(\eta \circ f)(a) - \frac{\alpha\varepsilon}{2}(\eta \circ f)(a') \\ &= \frac{\alpha}{2}(\eta \circ g)(y) + \frac{\alpha\varepsilon}{2}(\eta \circ g)(a) - \frac{\alpha\varepsilon}{2}(\eta \circ g)(a'). \end{aligned}$$

Define $z := (\eta \circ g)(a) - (\eta \circ f)(a) - (\eta \circ g)(a') + (\eta \circ f)(a') \in B$. This leads to $\frac{1}{4}((\eta \circ f)(y) - (\eta \circ g)(y)) = \varepsilon(\frac{1}{4}z)$ with $\|\frac{1}{4}z\| \leq 1$, thus $\frac{1}{4}z \in \widehat{\bigcirc}(B)$ and $\frac{1}{4}((\eta \circ f)(y) - (\eta \circ g)(y)) \in \widehat{\bigcirc}(B)$. This implies $\|\frac{1}{4}((\eta \circ f)(y) - (\eta \circ g)(y))\| = 0$ ([9], 6.1) in the t.c. space $\widehat{\bigcirc}(B)$ and thus $\frac{1}{4}((\eta \circ f)(y) - (\eta \circ g)(y)) = 0$ ([9], 6.9). So one gets $\eta(f(y)) = \eta(g(y))$ and, since η is injective, $f(y) = g(y)$ and $y \in \text{equ}(f, g)$. This shows $(\text{equ}(f, g))^\sim = \text{equ}(f, g)$. \square

In order to prove a characterization of epimorphisms in the category \mathbf{SC}_{sep} (resp. $\mathbf{Conv}_{\text{sep}}$) we note that each \mathbf{SC}_{sep} (resp. $\mathbf{Conv}_{\text{sep}}$)-morphism f has a factorization $f = g \circ h$ with a surjective morphism h and g the inclusion of the image of f in the respective category. Hence, in the following theorems we may restrict our attention to inclusions. \square

Let $D \in \mathbf{TC}(\mathbf{AC})$ and C be a subspace of D . C is called radially closed (r -closed) in D if and only if for all $x \in D$, $\alpha \in \bigcirc(K) \setminus \{0\}$, $\alpha x \in C$ implies $x \in C$ ([12], 1.1, 1.5). \widehat{C} denotes the smallest radially closed subspace of D

containing C ([12], 1.6). Now one can prove the following

2.7 Theorem: Let $D \in \mathbf{SC}_{\text{sep}}$, C be a subspace of D and $\text{in} : C \hookrightarrow D$ be the inclusion. Let $\eta' : D \longrightarrow \widehat{\bigcirc}(B)$ be the injective \mathbf{SC} -morphism from 2.4. Then $\eta'(C) - \eta'(C)$ is a separated t.c. subspace of $\eta'(D) - \eta'(D)$ (see 2.4). The following statements are equivalent:

(i) in is an \mathbf{SC}_{sep} -epimorphism.

(ii) $\widehat{\eta'(C) - \eta'(C)} = \eta'(D) - \eta'(D)$.

(iii) For every $y \in D$, $\varepsilon \in]0, 1[$, there exist $c, c' \in C$, $a, a' \in D$, $\alpha \in]0, 1[$ with

$$\frac{\alpha}{2}y + \frac{\alpha\varepsilon}{2}a + (1 - \frac{\alpha(1 + \varepsilon)}{2})c = \frac{\alpha\varepsilon}{2}a' + (1 - \frac{\alpha\varepsilon}{2})c'.$$

(iv) $\tilde{C} = D$.

Proof: Obviously, $\eta'(C) - \eta'(C)$ is a separated t.c. subspace of $\eta'(D) - \eta'(D)$.

(i) \implies (ii): A relation “ \sim ” is defined on D by $x \sim y$ if and only if $\eta'(x) - \eta'(y) \in \widehat{\eta'(C) - \eta'(C)}$ ($x, y \in D$) holds.

Obviously, “ \sim ” is reflexive and symmetric. Let $x, y, z \in D$ with $x \sim y$ and $y \sim z$. This implies $\frac{1}{2}(\eta'(x) - \eta'(z)) = \frac{1}{2}(\eta'(x) - \eta'(y)) + \frac{1}{2}(\eta'(y) - \eta'(z)) \in \widehat{\eta'(C) - \eta'(C)}$ and from [12], 1.5, we get $x \sim z$, and “ \sim ” is transitive. Let $x_i, y_i \in D$ with $x_i \sim y_i$ ($i \in \mathbb{N}$) and $\alpha \in \Omega_{sc}$. Then we compute in B

$$\begin{aligned} \eta'(\sum_i \alpha_i x_i) - \eta'(\sum_i \alpha_i y_i) &= \sum_i \alpha_i \eta'(x_i) - \sum_i \alpha_i \eta'(y_i) \\ &= \sum_i \alpha_i (\eta'(x_i) - \eta'(y_i)) \in \widehat{\eta'(C) - \eta'(C)}, \end{aligned}$$

hence $\sum_i \alpha_i x_i \sim \sum_i \alpha_i y_i$ holds, and “ \sim ” is a congruence relation on D . For all $x, y, z \in D$, $\beta \in]0, 1[$, $\beta x + (1 - \beta)z \sim \beta y + (1 - \beta)z$ implies $\beta(\eta'(x) - \eta'(y)) = \beta\eta'(x) + (1 - \beta)\eta'(z) - (\beta\eta'(y) + (1 - \beta)\eta'(z)) = \eta'(\beta x + (1 - \beta)z) - \eta'(\beta y + (1 - \beta)z) \in \widehat{\eta'(C) - \eta'(C)}$, hence, by [12], 1.5, $x \sim y$. Thus D/\sim is separated. Obviously, for all $c_0, c_1 \in C$ $c_0 \sim c_1$ holds. Let $\pi : D \longrightarrow D/\sim$

be the canonical projection and, for some $c_0 \in C$, $h : D \longrightarrow D/\sim$ be the constant mapping $h(x) := \pi(c_0)$ ($x \in D$). Then we have $\pi, h \in \mathbf{SC}_{\text{sep}}$ with $\pi \circ in = h \circ in$, and because in is an \mathbf{SC}_{sep} -epimorphism, $\pi = h$. This leads to $\pi(x) = h(x) = \pi(c_0)$ resp. $x \sim c_0$ ($x \in D$). Consequently, $x \sim y$ for all $x, y \in D$, hence $\eta'(x) - \eta'(y) \in \overbrace{\eta'(C) - \eta'(C)}^{\eta'(C) - \eta'(C)} (x, y \in D)$ holds and we have $\eta'(C) - \eta'(C) = \eta'(D) - \eta'(D)$.

(ii) \implies (iii): By [5], 2.5, for all $y \in D$, $\varepsilon \in]0, 1[$, $c_0 \in C$, there exist $c_1, c_2 \in C$, $a, a' \in D$, $\alpha \in]0, 1[$ with $\alpha(\eta'(y) - \eta'(c_0)) = \alpha(\varepsilon(\eta'(a') - \eta'(a))) + (1 - \alpha)(\eta'(c_1) - \eta'(c_2))$. Thus $\alpha\eta'(y) + \alpha\varepsilon\eta'(a) + (1 - \alpha)\eta'(c_2) = \alpha\varepsilon\eta'(a') + \alpha\eta'(c_0) + (1 - \alpha)\eta'(c_1)$ holds. Define

$$c := \frac{1-\alpha}{2-\alpha(1+\varepsilon)}c_2 + \frac{1-\alpha\varepsilon}{2-\alpha(1+\varepsilon)}c_0 \quad \text{and} \quad c' := \frac{1-\alpha}{2-\alpha\varepsilon}c_1 + \frac{1+\alpha(1-\varepsilon)}{2-\alpha\varepsilon}c_0.$$

Then $c, c' \in C$ and we get

$$\begin{aligned} & \eta' \left(\frac{\alpha}{2}y + \frac{\alpha\varepsilon}{2}a + (1 - \frac{\alpha(1+\varepsilon)}{2})c \right) = \frac{\alpha}{2}\eta'(y) + \frac{\alpha\varepsilon}{2}\eta'(a) + \frac{2-\alpha(1+\varepsilon)}{2}\eta'(c) \\ &= \frac{\alpha}{2}\eta'(y) + \frac{\alpha\varepsilon}{2}\eta'(a) + \frac{2-\alpha(1+\varepsilon)}{2} \left(\frac{1-\alpha}{2-\alpha(1+\varepsilon)}\eta'(c_2) + \frac{1-\alpha\varepsilon}{2-\alpha(1+\varepsilon)}\eta'(c_0) \right) \\ &= \frac{\alpha}{2}\eta'(y) + \frac{\alpha\varepsilon}{2}\eta'(a) + \frac{1-\alpha}{2}\eta'(c_2) + \frac{1-\alpha\varepsilon}{2}\eta'(c_0) \\ &= \frac{\alpha\varepsilon}{2}\eta'(a') + \frac{\alpha}{2}\eta'(c_0) + \frac{1-\alpha}{2}\eta'(c_1) + \frac{1-\alpha\varepsilon}{2}\eta'(c_0) \\ &= \frac{\alpha\varepsilon}{2}\eta'(a') + \frac{1-\alpha}{2}\eta'(c_1) + \frac{1+\alpha(1-\varepsilon)}{2}\eta'(c_0) \\ &= \frac{\alpha\varepsilon}{2}\eta'(a') + \frac{2-\alpha\varepsilon}{2} \left(\frac{1-\alpha}{2-\alpha\varepsilon}\eta'(c_1) + \frac{1+\alpha(1-\varepsilon)}{2-\alpha\varepsilon}\eta'(c_0) \right) \\ &= \frac{\alpha\varepsilon}{2}\eta'(a') + \frac{2-\alpha\varepsilon}{2}\eta'(c') = \eta' \left(\frac{\alpha\varepsilon}{2}a' + \frac{2-\alpha\varepsilon}{2}c' \right). \end{aligned}$$

Since η' is injective, we get

$$\frac{\alpha}{2}y + \frac{\alpha\varepsilon}{2}a + (1 - \frac{\alpha(1+\varepsilon)}{2})c = \frac{\alpha\varepsilon}{2}a' + \frac{2-\alpha\varepsilon}{2}c'.$$

(iii) \implies (iv): This follows trivially from 2.5.

(iv) \implies (i): This follows from 2.6. \square

§3 Epimorphisms of Separated Convex Spaces

By [2], 2.4, every separated convex space can be embedded in a separated a.c. space; therefore by a proof similar to 2.2 we get

3.1 Lemma: The forgetful functor $V : \mathbf{AC}_{\text{sep}} \rightarrow \mathbf{Conv}_{\text{sep}}$ has a faithful left adjoint and the unit of this adjunction is injective. \square

Denote by $\widehat{\bigcirc}_{fin}^{\text{sep}} : \mathbf{Vec}_1 \rightarrow \mathbf{AC}_{\text{sep}}$ the restriction of the comparsions functor $\widehat{\bigcirc}_{fin} : \mathbf{Vec}_1 \rightarrow \mathbf{AC}$ ([9], 3.6). $\widehat{\bigcirc}_{fin}$ has a left adjoint $S_{fin} : \mathbf{AC} \rightarrow \mathbf{Vec}_1$ ([9], 7.10). Thus the restriction $S_{fin}^{\text{sep}} : \mathbf{AC}_{\text{sep}} \rightarrow \mathbf{Vec}_1$ of S is a left adjoint of $\widehat{\bigcirc}_{fin}^{\text{sep}}$ with an injective unit ([9], 7.10, [10], p.1068). This yields the following

3.2 Corollary: The functor $V \circ \widehat{\bigcirc}_{fin}^{\text{sep}} : \mathbf{Vec}_1 \rightarrow \mathbf{Conv}_{\text{sep}}$ has a faithful left adjoint with an injective unit η . \square

If D is a separated convex space and, for $K := \mathbb{R}$, $\eta_D : D \rightarrow \widehat{\bigcirc}_{fin}(W)$, $W \in \mathbf{Vec}_1$, η_D as above or as in [2], 2.4, define $\eta' : D \rightarrow \widehat{\bigcirc}_{fin}(W)$ by $\eta'(x) := \frac{1}{2}\eta_D(x)$ ($x \in D$). η' is an injective \mathbf{Conv} -morphism. Now, consider the subset of $\widehat{\bigcirc}_{fin}(W)$, $\eta'(D) - \eta'(D) := \{\eta'(x) - \eta'(y) \mid x, y \in D\}$. $\widehat{\bigcirc}_{fin}(W)$ is a real separated a.c. space and we have by a proof analogous to that of 2.4 the following

3.3 Lemma: The above subset $\eta'(D) - \eta'(D) \subset \widehat{\bigcirc}_{fin}(W)$ is a real separated a.c. subspace of $\widehat{\bigcirc}_{fin}(W)$. \square

In the proof of 2.6 only the convex structure of D and E was used, hence, we actually also proved the

3.4 Proposition: Let $f, g : D \rightarrow E$ be \mathbf{Conv} -morphisms and E be separated. Then the equalizer $\text{equ}(f, g) := \{x \in D \mid f(x) = g(x)\}$ is a subspace of D and $(\text{equ}(f, g))^\sim = \text{equ}(f, g)$ (see 2.5) holds. \square

For $D \in \mathbf{AC}$ a mapping $d : D \times D \rightarrow \mathbb{O}(\mathbb{R})$ is defined by $d((x, y)) := \|\frac{1}{2}x - \frac{1}{2}y\|$ ($x, y \in D$). d is called the distance function on D ([12], 3.1). For $D \in \mathbf{AC}$ and a subspace C of D , a mapping $d(-, C) : D \rightarrow \mathbb{O}(\mathbb{R})$ is defined by $d(-, C)(x) := d(x, C) := \inf\{d((x, y)) | y \in C\}$ ($x \in D$). C is said to be d -closed if and only if for any $x \in D$ $d(x, C) = 0$ implies $x \in C$ ([12], 3.3). C is said to be closed if and only if it is both, d -closed and r -closed. Finally, for $D \in \mathbf{AC}$ with a subspace C , let \tilde{C} be the smallest closed subspace of D containing C ([12], 3.14). \tilde{C} is said to be the closed hull of C in D . Now we can prove the following

3.5 Theorem: Let $D \in \mathbf{Conv}_{\text{sep}}$, C be a subspace of D and $in : C \hookrightarrow D$ be the inclusion. Let $\eta' : D \rightarrow \widehat{\mathbb{O}}_{fin}(W)$ be the injective \mathbf{Conv} -morphism from 3.3. Then $\eta'(C) - \eta'(C)$ is a real separated a.c. subspace of $\eta'(D) - \eta'(D)$. The following statements are equivalent :

(i) in is an $\mathbf{Conv}_{\text{sep}}$ -epimorphism.

(ii) $\widehat{\eta'(C) - \eta'(C)} = \eta'(D) - \eta'(D)$.

(iii) For every $y \in D$, $\varepsilon \in]0, 1[$, there exist $c, c' \in C$, $a, a' \in D$, $\alpha \in]0, 1[$ with

$$\frac{\alpha}{2}y + \frac{\alpha\varepsilon}{2}a + \left(1 - \frac{\alpha(1+\varepsilon)}{2}\right)c = \frac{\alpha\varepsilon}{2}a' + \left(1 - \frac{\alpha\varepsilon}{2}\right)c'.$$

(iv) $\tilde{C} = D$.

Proof: Obviously, $\eta'(C) - \eta'(C)$ is a real separated a.c. subspace of $\eta'(D) - \eta'(D)$ ([10], 11.14).

(i) \Rightarrow (ii): A relation “~” is defined on D by $x \sim y$ if and only if $\eta'(x) - \eta'(y) \in \widehat{\eta'(C) - \eta'(C)}$ ($x, y \in D$) holds. In the same way as in 2.7 one sees, that “~” is a congruence relation on D .

Furthermore, D/\sim is separated. Indeed, let $u, v, x_n, y_n \in D$ with $(1 - \frac{1}{n})u +$

$\frac{1}{n}x_n \sim (1 - \frac{1}{n})v + \frac{1}{n}y_n$ ($n \in \mathbb{N}$). By definition, this implies $(1 - \frac{1}{n})(\eta'(u) - \eta'(v)) + \frac{1}{n}(\eta'(x_n) - \eta'(y_n)) = \eta'((1 - \frac{1}{n})u + \frac{1}{n}x_n) - \eta'((1 - \frac{1}{n})v + \frac{1}{n}y_n) \in \underline{\eta'(C)} - \eta'(C)$ ($n \in \mathbb{N}$) and for all $n \in \mathbb{N}$, $n > 1$, we get $\frac{1}{2}(\eta'(u) - \eta'(v)) - \frac{1}{2}\frac{1}{n-1}(\eta'(y_n) - \eta'(x_n)) \in \underline{\eta'(C)} - \eta'(C)$. Let $\varepsilon \in]0, 1[$. Then there exists a $n \in \mathbb{N}$, $n > 1$, with $\frac{1}{n-1} < \varepsilon$. From $\|\frac{1}{n-1}(\eta'(y_n) - \eta'(x_n))\| \leq \frac{1}{n-1} < \varepsilon$ we conclude $\eta'(u) - \eta'(v) \in \underline{\eta'(C)} - \eta'(C)$ ([12], 3.15), thus $u \sim v$, and D/\sim is separated. Obviously, for all $c_0, c_1 \in C$, $c_0 \sim c_1$ holds. Now, in the same way as in the proof of 2.7, $\underline{\eta'(C)} - \eta'(C) = \eta'(D) - \eta'(D)$ follows.

(ii) \Rightarrow (iii): By [5], 3.1 the proof consists of the same computations as the proof of (ii) \Rightarrow (iii) in 2.7.

(iii) \Rightarrow (iv): This follows from the definition of \tilde{C} in 2.5.

(iv) \Rightarrow (i): This follows from 3.4. \square

As \mathbf{SC}_{sep} is a subcategory of $\mathbf{Conv}_{\text{sep}}$, 2.7 and 3.5 imply the

3.6 Corollary: Let $f : C \rightarrow D$ be an \mathbf{SC}_{sep} -morphism. Then f is an \mathbf{SC}_{sep} -epimorphism if and only if f is an $\mathbf{Conv}_{\text{sep}}$ -epimorphism. \square

§4 Epimorphisms of Preseparated Convex Spaces

4.1 Theorem: Let D be a preseparated convex space, and C be a subspace of D . Let $in : C \hookrightarrow D$ be the inclusion. Equivalent are:

(i) in is an $\mathbf{Conv}_{\text{psep}}$ -epimorphism.

(ii) For all $y \in D$ there exist $c \in C$ and $\alpha \in]0, 1]$ with $\alpha y + (1 - \alpha)c \in C$.

Proof: (i) \Rightarrow (ii): By 1.1 the relation “ \sim ” on D , defined by $x \sim y$ if and only if there exist $c, c' \in C$, $\alpha \in]0, 1]$ with $\alpha x + (1 - \alpha)c = \alpha y + (1 - \alpha)c'$ ($x, y \in D$) is a congruence relation on D . Let $x, y, z \in D$, $\alpha \in]0, 1[$ with $\alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z$. Thus there are $c, c' \in C$, $\beta \in]0, 1]$ with

$\beta(\alpha x + (1 - \alpha)z) + (1 - \beta)c = \beta(\alpha y + (1 - \alpha)z) + (1 - \beta)c'$. Put

$$a := \frac{\beta\alpha}{1-\beta(1-\alpha)}x + \frac{1-\beta}{1-\beta(1-\alpha)}c, \quad b := \frac{\beta\alpha}{1-\beta(1-\alpha)}y + \frac{1-\beta}{1-\beta(1-\alpha)}c'.$$

Then $(\beta(1 - \alpha))z + (1 - \beta(1 - \alpha))a = (\beta(1 - \alpha))z + (\beta\alpha)x + (1 - \beta)c = \beta(\alpha x + (1 - \alpha)z) + (1 - \beta)c = \beta(\alpha y + (1 - \alpha)z) + (1 - \beta)c' = (\beta(1 - \alpha))z + (\beta\alpha)y + (1 - \beta)c' = (\beta(1 - \alpha))z + (1 - \beta(1 - \alpha))b$ holds. Since D is preseparated and $0 < 1 - \beta(1 - \alpha)$ holds, $a = b$ and by definition, $x \sim y$ follows. Thus D/\sim is preseparated. Obviously, for all $c, c' \in C$, $c \sim c'$ holds. Let $c_0 \in C$ and $\pi : D \rightarrow D/\sim$ be the canonical projection and $f : D \rightarrow D/\sim$ be the constant mapping $f(x) := \pi(c_0)$ ($x \in D$). Then π, f are $\mathbf{Conv}_{\text{psep}}$ -morphisms with $\pi \circ in = f \circ in$, which implies $\pi = f$, because in is a $\mathbf{Conv}_{\text{psep}}$ -epimorphism, thus $\pi(y) = f(y) = \pi(c_0)$ for all $y \in D$. This implies the existence of elements $c_1, c_2 \in C$, $\gamma \in]0, 1]$ with $\gamma y + (1 - \gamma)c_1 = \gamma c_0 + (1 - \gamma)c_1 \in C$ and (ii) is fulfilled.

(ii) \Rightarrow (i): Let $f, g : D \rightarrow E$ be $\mathbf{Conv}_{\text{psep}}$ -morphisms with $f \circ in = g \circ in$. For $y \in D$ there are, because of (ii), $c \in C$, $\beta \in]0, 1]$ with $\beta y + (1 - \beta)c \in C$. This implies $\beta f(y) + (1 - \beta)f(c) = f(\beta y + (1 - \beta)c) = g(\beta y + (1 - \beta)c) = \beta g(y) + (1 - \beta)g(c) = \beta g(y) + (1 - \beta)f(c)$. Hence, we have $f(y) = g(y)$, since E is preseparated. Consequently, $f = g$ and in is a $\mathbf{Conv}_{\text{psep}}$ -epimorphism. \square

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p-Banach Spaces and p-Totally Convex Spaces I

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Introduction

In [8] Pumplün and Röhrl introduced the category \mathbf{TC} (resp. \mathbf{TC}_{fin}) of (finitely) totally convex (t.c.) spaces, which are the Eilenberg-Moore algebras of the monad induced by the unit ball functor from the category of Banach spaces (resp. normed vector spaces) with linear contractions to the category of sets. In accordance with [2] we use the term “absolutely convex” for the spaces Pumplün and Röhrl call “finitely totally convex”. \mathbf{AC} denotes the category of absolutely convex spaces.

In the present paper the categories \mathbf{TC}_p of p-totally convex and \mathbf{AC}_p of p-absolutely convex spaces are introduced and investigated ($0 < p \leq 1$). For $p = 1$ they coincide with the categories \mathbf{TC} and \mathbf{AC} and the results proved by Pumplün and Röhrl ([8], [9]) are contained as a special case in this paper, resp. in a continuation of this. The notion of p-absolutely convex spaces is a generalization of p-absolutely convex subsets of \mathbb{IK} -vector spaces and p-totally convex spaces are a generalization of p-totally convex subsets of topological \mathbb{IK} -vector spaces, $\mathbb{IK} = \mathbb{IR}, \mathbb{C}$.

In §1 we introduce the categories \mathbf{Vec}_p and \mathbf{Ban}_p of p-normed vector spaces and p-Banach spaces [4] over the field $\mathbb{IK} = \mathbb{IR}, \mathbb{C}$. $\bigcirc_p : \mathbf{Ban}_p \rightarrow \mathbf{Set}$ (resp. $\bigcirc_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{Set}$) denotes the canonical unit ball functor. \bigcirc_p and $\bigcirc_{p,fin}$ have left adjoints l_p resp. $l_{p,fin}$. Here, for a set X , $l_p(X)$ is the usual l_p -space on X ([4]), while $l_{p,fin}(X)$ is the subspace of $l_p(X)$ consisting of all functions of finite support. \bigcirc_p and $\bigcirc_{p,fin}$ are premonadic ([10]), i.e. the comparison functor from \mathbf{Ban}_p resp. \mathbf{Vec}_p to the corresponding category of Eilenberg-Moore algebras is full and faithful.

After the definition of p-totally and p-absolutely convex spaces and the proof of some computational rules in §2 it is shown in §3 that the categories \mathbf{TC}_p and \mathbf{AC}_p are the Eilenberg-Moore algebras for \bigcirc_p resp. $\bigcirc_{p,fin}$. Both categories possess an internal hom-functor and a tensor product, they are closed categories in the sense

of [7] and even autonomous categories in the sense of Linton ([6]). In §4 a semi p-norm on p-absolutely (resp. p-totally) convex spaces is introduced and properties of congruence relations on these spaces are studied. There are many differences between totally (resp. absolutely) and p-totally (resp. p-absolutely) convex spaces; this is especially true for the central Theorem 4.10, for which R.Börger gave an important hint. \square

§1 p-Normed Vector Spaces and p-Banach Spaces

p will always denote a real number with $0 < p \leq 1$, and one puts $\omega_p := (\frac{1}{2})^{\frac{1}{p}-1}$; \mathbb{N} is the set of natural numbers and \mathbb{K} is the field \mathbb{R} or \mathbb{C} of real or complex numbers. For a set M we put $|M| := \text{card}(M)$. \square

1.1 Definition: Let $(V_i, \| \cdot \|_i)$ ($i = 0, 1$) be p-normed vector spaces ([4], p.115). A \mathbb{K} -linear mapping $f : V_0 \rightarrow V_1$ is called a contraction if and only if for every $x \in V_0$ $\|f(x)\|_1 \leq \|x\|_0$ holds. $\mathbf{Vec}_{\mathbb{K}, p}$ or briefly \mathbf{Vec}_p is the category of all p-normed vector spaces together with these contractions. \square

A p-Banach space is a complete p-normed vector space ([4], p.116). The full subcategory of \mathbf{Vec}_p which is determined by all p-Banach spaces is denoted by \mathbf{Ban}_p . It is obvious that for all p, s with $0 < s \leq p$ \mathbf{Vec}_p (resp. \mathbf{Ban}_p) is a subcategory of \mathbf{Vec}_s (resp. \mathbf{Ban}_s). \square

All topological statements will refer to the topology induced by the p-norm. For a bounded linear mapping $f : V \rightarrow W$ between p-normed vector spaces $\|f\| := \sup \left\{ \frac{\|f(x)\|}{\|x\|} \mid x \in V \setminus \{0\} \right\} = \sup \{ \|f(x)\| \mid \|x\| \leq 1 \} = \inf \{ M \geq 0 \mid \|f(x)\| \leq M\|x\| \ (x \in V) \}$ denotes the usual norm. Every $(V, \| \cdot \|) \in \mathbf{Vec}_p$ with respect to the supremums p-norm on $V \times V$ is a locally p-convex space ([4], 6.5.1) and the p-norm $\| \cdot \| : V \rightarrow \mathbb{R}$ is continuous. \square

1.2 Proposition: For a p-normed vector space $(V, \| \cdot \|)$ and a closed subspace U the canonical projection $\pi : V \rightarrow V/U$ satisfies :

(i) π is a coequalizer in \mathbf{Vec}_p . (ii) $\|\pi\| = 0$ or $\|\pi\| = 1$ holds.

Proof: (i) Obviously, $W := \{(x, y) \in V \times V \mid x - y \in U\}$ is a p-normed subspace of $V \times V$ with respect to the supremums p-norm on $V \times V$. Apparently, π is the coequalizer of the restrictions of the canonical projections $\pi_1, \pi_2 : V \times V \rightarrow V$ to W in \mathbf{Vec}_p .

(ii) Let $\|\pi\| \neq 0$ and $\|\cdot\|'$ denote the p-norm on V/U . Obviously, $\|\pi\|^{-1}\|\cdot\|'$ is a p-norm on V/U and the canonical projection $\hat{\pi} : (V, \|\cdot\|) \rightarrow (V/U, \|\pi\|^{-1}\|\cdot\|')$ is a \mathbf{Vec}_p -morphism. The identical mapping $\varphi : (V/U, \|\cdot\|') \rightarrow (V/U, \|\pi\|^{-1}\|\cdot\|')$ is \mathbb{K} -linear and $\varphi \circ \pi = \hat{\pi}$ holds. Obviously, φ is a \mathbf{Vec}_p -morphism. Because of $\|\pi\| \neq 0$ there exists a $v \in V \setminus U$ and $\|\pi\|^{-1}\|v + U\|' = \|\pi\|^{-1}\|\varphi(v + U)\|' \leq \|v + U\|'$ implies $\|\pi\| = 1$. \square

For a set mapping $g : X \rightarrow \mathbb{K}$ put $\text{supp } g := \{x \in X \mid g(x) \neq 0\}$. For a set I and real numbers $\alpha_i \geq 0$ ($i \in I$), $\sum_{i \in I} \alpha_i < \infty$ always implies that $\text{supp}\{\alpha_i \mid i \in I\} := \{i \in I \mid \alpha_i \neq 0\}$ is countable. For a set X define $l_p(X) := \{g \in \mathbf{Set}(X, \mathbb{K}) \mid \text{supp } g \text{ is countable and } \sum_{x \in X} |g(x)|^p < \infty\}$. The p-norm $\|\cdot\| : l_p(X) \rightarrow \mathbb{R}$ is defined by $\|g\| := (\sum_{x \in X} |g(x)|^p)^{\frac{1}{p}}$ ($g \in l_p(X)$). For $x \in X$, $\delta^x : X \rightarrow \mathbb{K}$ is the Dirac function at x , that is $\delta^x(x) = 1$ and $\delta^x(y) = 0$ for $y \neq x$ ($y \in X$).

1.3 Proposition ([4], p.118): For every set X $l_p(X)$ is a p-Banach space and $(\delta^x)_{x \in X}$ is a Schauder-basis ([4], p.292) of $l_p(X)$. \square

The unit ball functor $\bigcirc_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{Set}$ is induced by the mapping assigning to each p-normed vector space V its closed unit ball $\bigcirc_{p,fin}(V) := \{x \in V \mid \|x\| \leq 1\}$. Its restriction to \mathbf{Ban}_p is denoted by \bigcirc_p . If no misunderstandings are possible, one simply writes $\bigcirc_p : \mathbf{Vec}_p \rightarrow \mathbf{Set}$ instead of $\bigcirc_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{Set}$. Especially one has for $\mathbb{K} \in \mathbf{Ban}_p$ $\bigcirc_p(\mathbb{K}) = \bigcirc_1(\mathbb{K}) = \{\alpha \in \mathbb{K} \mid |\alpha| \leq 1\}$. \square

1.4 Proposition (cf. [8], 1.1): The functor $\bigcirc_p : \mathbf{Ban}_p \rightarrow \mathbf{Set}$ has l_p as left adjoint, is premonadic, but fails to be monadic ([10], 2.10).

Proof: Let X be a set. The mapping $\eta_X : X \rightarrow \bigcirc_p \circ l_p(X)$ is defined by $\eta_X(x) :=$

δ^x ($x \in X$). For a set mapping $\psi : X \rightarrow \bigcirc_p(B)$ obviously a \mathbf{Ban}_p -morphism $\hat{\psi} : l_p(X) \rightarrow B$ with $\bigcirc_p(\hat{\psi}) \circ \eta_X = \psi$ is uniquely determined. $\hat{\psi} : l_p(X) \rightarrow B$ defined by $\hat{\psi}(g) := \sum_{x \in X} g(x)\psi(x)$ is a (well defined) \mathbf{Ban}_p -morphism with $\bigcirc_p(\hat{\psi}) \circ \eta_X = \psi$, rendering the left adjoint in question. The counit ε of this adjunction is uniquely determined by the equation $\varepsilon_B(\delta^x) = x$ ($x \in \bigcirc_p(B)$, $B \in \mathbf{Ban}_p$). Since, for $B \in \mathbf{Ban}_p$ and any $x \in B \setminus \{0\}$, $x = \|x\|\varepsilon_B(\delta^{\frac{x}{\|x\|}}) = \varepsilon_B(\|x\|\delta^{\frac{x}{\|x\|}})$ and $\varepsilon_B(0) = 0$ holds, $\varepsilon_B : l_p \circ \bigcirc_p(B) \rightarrow B$ is a surjection ($B \in \mathbf{Ban}_p$). $\text{Ker}\varepsilon_B := \varepsilon_B^{-1}(\{0_B\})$ and $l_p \circ \bigcirc_p(B)/\text{Ker}\varepsilon_B$ are complete. $\pi_B : l_p \circ \bigcirc_p(B) \rightarrow l_p \circ \bigcirc_p(B)/\text{Ker}\varepsilon_B$ denotes the canonical projection. Thus there exist a \mathbb{IK} -linear isomorphism $\varphi : l_p \circ \bigcirc_p(B)/\text{Ker}\varepsilon_B \rightarrow B$ with $\varphi \circ \pi_B = \varepsilon_B$. By an elementary computation, φ is a \mathbf{Ban}_p -isomorphism. By 1.2(i), π_B is a coequalizer in \mathbf{Vec}_p , and thus in \mathbf{Ban}_p . $\varphi \circ \pi_B = \varepsilon_B$ implies that ε_B is a coequalizer in \mathbf{Ban}_p ($B \in \mathbf{Ban}_p$). Consequently, \bigcirc_p is premonadic. That \bigcirc_p fails to be monadic follows by [8], 4.6. \square

For any set X put $l_{p,fin}(X) := \mathbb{IK}^{(X)}$ and a p -norm $\|\cdot\| : l_{p,fin}(X) \rightarrow \mathbb{IR}$ is defined by $\|g\| := (\sum_{x \in X} |g(x)|^p)^{\frac{1}{p}}$ ($g \in l_{p,fin}(X)$). Obviously we have

1.5 Proposition: For every set X ($l_{p,fin}(X), \|\cdot\|$) is a p -normed vector space and the family $(\delta^x)_{x \in X}$ is a basis of $l_{p,fin}(X)$. \square

Similar to 1.4, resp. to the case $p = 1$ one proves the following two propositions:

1.6 Proposition: The functor $\bigcirc_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{Set}$ possesses $l_{p,fin}$ as left adjoint. $\bigcirc_{p,fin}$ is premonadic and fails to be monadic. \square

1.7 Proposition: \mathbf{Ban}_p is a reflective subcategory of \mathbf{Vec}_p . The reflection morphism is a dense embedding. \square

1.8 Lemma: The canonical forgetful functor $V : \mathbf{Vec}_p \rightarrow \mathbf{Set}$ (resp. $V : \mathbf{Ban}_p \rightarrow \mathbf{Set}$) has no left adjoint.

Proof: Assume, $V : \mathbf{Ban}_p \rightarrow \mathbf{Set}$ has a left adjoint $F : \mathbf{Set} \rightarrow \mathbf{Ban}_p$ with unit $\eta : \mathbf{Set} \rightarrow V \circ F$. Define the mapping $f : \mathbb{IN} \rightarrow V(\mathbb{IK})$ by $f(n) := n\|\eta_{\mathbb{IN}}(n)\| + 1$

($n \in \mathbb{N}$); here $\| \cdot \|$ means the p-norm on $F(\mathbb{N})$. Then a simple computation leads to a contradiction. \square

The category \mathbf{Vec}_p has products, namely $V = \{v = (v_i)_{i \in I} \in X_{i \in I} V_i \mid \sup\{\|v_i\| \mid i \in I\} < \infty\}$ for $V_i \in \mathbf{Vec}_p$ ($i \in I$) with $\|v\| := \sup\{\|v_i\| \mid i \in I\}$ for $v \in V$, and equalizers hence it is complete. \mathbf{Ban}_p as a full, reflective subcategory is also complete and products and equalizers in \mathbf{Ban}_p are the same as in \mathbf{Vec}_p . For categorical reasons the cocompleteness of \mathbf{Vec}_p and \mathbf{Ban}_p results from the completeness but for later applications the coproduct and the coequalizers in \mathbf{Vec}_p and \mathbf{Ban}_p will be explicitly constructed now. \square

Let $(V_i)_{i \in I}$ be a family of p-normed vector spaces and put $V := \{v = (v_i)_{i \in I} \in X_{i \in I} V_i \mid |\text{supp } v| < \infty\}$. For $v \in V$ put $\|v\| := (\sum_{i \in \text{supp } v} \|v_i\|^p)^{\frac{1}{p}}$. The mapping $\mu_i : V_i \rightarrow V$ is defined by $(\mu_i(x))_i := x$ and $(\mu_i(x))_j := 0$ for $j \neq i$ ($i, j \in I$).

1.9 Proposition: $(V, \mu_i : V_i \rightarrow V)_{i \in I}$ is a coproduct of $(V_i)_{i \in I}$ in \mathbf{Vec}_p .

Proof: V becomes a \mathbb{K} -vector space under pointwise addition and scalar-multiplication. Furthermore, $(V, \| \cdot \|)$ is a p-normed vector space. The mapping $\mu_i : V_i \rightarrow V$ is \mathbb{K} -linear, injective and because of $\|\mu_i(x)\| = \|x\|$ a \mathbf{Vec}_p -morphism ($i \in I$).

Let $(f_i : V_i \rightarrow W)_{i \in I}$ be a family of \mathbf{Vec}_p -morphisms. A \mathbf{Vec}_p -morphism $f : V \rightarrow W$ with $f \circ \mu_i = f_i$ ($i \in I$) is uniquely determined, since for all $v \in V$, $v = \sum_{i \in \text{supp } v} \mu_i(v_i)$ holds. Define the mapping $f : V \rightarrow W$ by $f(v) := \sum_{i \in \text{supp } v} f_i(v_i)$. Obviously, f is \mathbb{K} -linear and $f \circ \mu_i = f_i$ ($i \in I$) is fulfilled. Because of $\|f(v)\|^p = \|\sum_{i \in \text{supp } v} f_i(v_i)\|^p \leq \sum_{i \in \text{supp } v} \|f_i(v_i)\|^p \leq \sum_{i \in \text{supp } v} \|v_i\|^p = \|v\|^p$, f is a \mathbf{Vec}_p -morphism. \square

For \mathbf{Vec}_p -morphisms $f, g : V \rightarrow W$ define $U := \overline{(f - g)(V)}$, i.e. the closure of the subspace $(f - g)(V)$ of W and let $\pi : W \rightarrow W/U$ be the canonical projection. Then we have the following

1.10 Proposition: $\pi : W \rightarrow W/U$ is a coequalizer of f and g in \mathbf{Vec}_p .

Proof: By 1.2, $\pi : W \rightarrow W/U$ is a \mathbf{Vec}_p -morphism. Obviously, $\pi \circ f = \pi \circ g$ holds and for every \mathbf{Vec}_p -morphism $\tau : W \rightarrow Z$ with $\tau \circ f = \tau \circ g$ the continuity of τ implies $\tau(U) = \{0\}$. Consequently, the mapping $\varphi : W/U \rightarrow Z$, defined by $\varphi(w + U) := \tau(w)$ ($w \in W$) is well defined. φ is a \mathbf{Vec}_p -morphism fulfilling $\varphi \circ \pi = \tau$ and is uniquely determined by this equation. \square

Let $(B_i)_{i \in I}$ be a family of p-Banach spaces and define $B := \{b = (b_i)_{i \in I} \in X_{i \in I} B_i \mid \sum_{i \in I} \|b_i\|^p < \infty\}$. For $b \in B$ put $\|b\| := (\sum_{i \in I} \|b_i\|^p)^{\frac{1}{p}}$. Define the mapping $\mu_i : B_i \rightarrow B$ by $(\mu_i(x))_i := x$ and $(\mu_i(x))_j := 0$ for $j \neq i$ ($i, j \in I$). Then we have

1.11 Proposition: $(B, \mu_i : B_i \rightarrow B)_{i \in I}$ is a coproduct of $(B_i)_{i \in I}$ in \mathbf{Ban}_p .

Proof: Denote by C the coproduct of $(B_i)_{i \in I}$ in \mathbf{Vec}_p . By $l_p \simeq Q \circ l_{p,fin}$ (where $Q : \mathbf{Vec}_p \rightarrow \mathbf{Ban}_p$ is the completion functor (1.7)), B is the completion of C in \mathbf{Ban}_p . Now, by 1.7, we are finished. \square

Let $f, g : B \rightarrow B'$ be \mathbf{Ban}_p -morphisms and $U := \overline{(f - g)(B)}$. Then B'/U is complete and one gets for the canonical projection $\pi : B' \rightarrow B'/U$

1.12 Proposition: $\pi : B' \rightarrow B'/U$ is a coequalizer of f and g in \mathbf{Ban}_p . \square

1.13 Definition: For $V, W \in \mathbf{Vec}_p$ ($V, W \in \mathbf{Ban}_p$) put $\text{Hom}(V, W) := \{f : V \rightarrow W \mid f \text{ is } \mathbb{K}\text{-linear and bounded}\}$. \square

For $V, W \in \mathbf{Vec}_p$, $f \in \text{Hom}(V, W)$ by definition $\|f\| = \inf\{M \geq 0 \mid \|f(x)\| \leq M\|x\| \text{ for all } x \in V\}$ holds. Obviously, $(\text{Hom}(V, W), \|\cdot\|)$ is canonically a p-normed vector space. For $B, B' \in \mathbf{Ban}_p$ by a straightforward computation ([4], 3.2.3) $(\text{Hom}(B, B'), \|\cdot\|)$ is a p-Banach space. Here we have the following

1.14 Proposition: The functors $\text{Hom}(-, -) : \mathbf{Vec}_p \times \mathbf{Vec}_p \rightarrow \mathbf{Vec}_p$ (resp. $\text{Hom}(-, -) : \mathbf{Ban}_p \times \mathbf{Ban}_p \rightarrow \mathbf{Ban}_p$) are internal ([6], p. 637) with respect to $\bigcirc_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{Set}$ (resp. $\bigcirc_p : \mathbf{Ban}_p \rightarrow \mathbf{Set}$). \square

A \mathbb{K} -bilinear mapping $\psi : V \times W \rightarrow Z$ ($V, W, Z \in \mathbf{Vec}_p$) is called bounded if and only if there exists a $\lambda \geq 0$ with $\|\psi((v, w))\| \leq \lambda \|v\| \|w\|$ for all $v \in V, w \in W$. In this case one puts $\|\psi\| := \sup \left\{ \frac{\|\psi((v, w))\|}{\|v\| \|w\|} \mid v \in V, w \in W, \|v\| \|w\| \neq 0 \right\}$, and in case $\|\psi\| \leq 1$ ψ is called a bilinear contraction. The following definition is the natural generalization of the usual notation of the projective tensor product to the case $p < 1$.

1.15 Definition: Let $V, W \in \mathbf{Vec}_p(\mathbf{Ban}_p)$. A triple $(E, \otimes : V \times W \rightarrow E)$ is called a projective tensor product of V and W in $\mathbf{Vec}_p(\mathbf{Ban}_p)$, if $E \in \mathbf{Vec}_p(\mathbf{Ban}_p)$, $\otimes : V \times W \rightarrow E$ is a bilinear contraction and for every bounded bilinear mapping $\psi : V \times W \rightarrow Z$ ($Z \in \mathbf{Vec}_p(\mathbf{Ban}_p)$) there exists a uniquely determined bounded \mathbb{K} -linear mapping $\varphi : E \rightarrow Z$ with $\varphi \circ \otimes = \psi$. \square

1.16 Proposition: Let $V, W \in \mathbf{Vec}_p$. Then there exists a projective tensor product of V and W in \mathbf{Vec}_p uniquely determined up to isomorphism, denoted by $V \otimes W$. The functor $- \otimes -$ is left adjoint to $\text{Hom}(-, -)$.

Proof: For $V, W \in \mathbf{Vec}_p$ define a mapping $\sigma : V \times W \rightarrow l_{p,fin}(V \times W)$ by $\sigma((v, w)) := \|v\| \|w\| \delta^{(v,w)}$ ($v \in V, w \in W$). Let U be the closed subspace of $l_{p,fin}(V \times W)$ spanned by all elements $\sigma((v+v', w)) - \sigma((v, w)) - \sigma((v', w))$, $\sigma((v, w+w')) - \sigma((v, w)) - \sigma((v, w'))$, $\sigma((\alpha v, w)) - \alpha \sigma((v, w))$, $\sigma((v, \alpha w)) - \alpha \sigma((v, w))$ ($v, v' \in V, w, w' \in W, \alpha \in \mathbb{K}$). By 1.2, the canonical projection $\pi : l_{p,fin}(V \times W) \rightarrow l_{p,fin}(V \times W)/U$ is a \mathbf{Vec}_p -morphism. Put $V \otimes W := l_{p,fin}(V \times W)/U$. The mapping $\otimes := \pi \circ \sigma$ is \mathbb{K} -bilinear and obviously a contraction. Now, by a straightforward computation the assertion follows. \square

For p -Banach spaces B and C put $B \hat{\otimes} C := Q(B \otimes C)$ (where Q is the completion functor). Let $in : B \otimes C \hookrightarrow B \hat{\otimes} C$ be the isometric inclusion and put $\hat{\otimes} := in \circ \otimes : B \times C \hookrightarrow B \hat{\otimes} C$. Obviously we have

1.17 Proposition: For $B, C \in \mathbf{Ban}_p$, $(B \hat{\otimes} C, \hat{\otimes})$ is a projective tensor product of B and C in \mathbf{Ban}_p . The induced functor $- \hat{\otimes} -$ is left adjoint to $\text{Hom}(-, -)$. \square

It is well known, that \mathbb{IK} is a (single object) cogenerator in the categories \mathbf{Vec}_1 and \mathbf{Ban}_1 . For $p < 1$, \mathbb{IK} is not a cogenerator in \mathbf{Vec}_p (\mathbf{Ban}_p), which is shown by the following example.

Let $p < 1$ and $L_p([0, 1]) := \{f : [0, 1] \rightarrow \mathbb{IK} \mid f \text{ is Borel-measurable and } \int_0^1 |f(x)|^p dx < \infty\}$. Obviously, under pointwise addition and scalar-multiplication, $L_p([0, 1])$ is a \mathbb{IK} -vector subspace of $\mathbb{IK}^{[0,1]}$ and one defines $M_p := \{f \in L_p([0, 1]) \mid f = 0 \text{ almost everywhere}\}$. M_p is a \mathbb{IK} -vector subspace of $L_p([0, 1])$, and one puts $\mathcal{L}_p([0, 1]) := L_p([0, 1])/M_p$. Let $\pi : L_p([0, 1]) \rightarrow \mathcal{L}_p([0, 1])$ be the canonical projection and define $\|\pi(f)\| := \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}$ ($f \in L_p([0, 1])$). Then one has

1.18 Proposition ([5], p. 158): $\mathcal{L}_p([0, 1])$ is a p -Banach space and the zero morphism is the only \mathbb{IK} -linear continuous mapping $\mathcal{L}_p([0, 1]) \rightarrow \mathbb{IK}$. \square

§2 p-Totally and p-Absolutely Convex Spaces

Put $\Omega_p := \bigcirc_p \circ l_p(\mathbb{N}) = \{\alpha = (\alpha_i)_{i \in \mathbb{N}} \in \mathbb{IK}^{\mathbb{N}} \mid \sum_i |\alpha_i|^p \leq 1\}$. For $\alpha \in \Omega_p$ define $supp\alpha := \{i \in \mathbb{N} \mid \alpha_i \neq 0\}$. Finally, put $\Omega_{p,fin} := \bigcirc_{p,fin} \circ l_{p,fin}(\mathbb{N}) = \{\alpha \in \Omega_p \mid |supp\alpha| < \infty\}$. By a routine computation one gets

2.1 Proposition: Let $\alpha = (\alpha_i)_{i \in \mathbb{N}} \in \Omega_p$ ($\Omega_{p,fin}$). $\beta^i = (\beta_j^i)_{j \in \mathbb{N}} \in \Omega_p$ ($\Omega_{p,fin}$) ($i \in \mathbb{N}$). Then $\left(\sum_i \alpha_i \beta_j^i\right)_{j \in \mathbb{N}} \in \Omega_p$ ($\Omega_{p,fin}$) holds. \square

2.2 Definition: An Ω_p -algebra is a set X together with a map $\Omega_p \rightarrow \mathbf{Set}(X^{\mathbb{N}}, X)$, $\alpha \mapsto \alpha_X$ ($\alpha \in \Omega_p$). A morphism from the Ω_p -algebra X to the Ω_p -algebra Y is a set mapping $f : X \rightarrow Y$ satisfying for any $\alpha \in \Omega_p$, $f \circ \alpha_X = \alpha_Y \circ f^{\mathbb{N}}$, where $f^{\mathbb{N}} : X^{\mathbb{N}} \rightarrow Y^{\mathbb{N}}$ is defined componentwise.

Obviously, the Ω_p -algebras together with their morphisms form a category, the composition of the morphisms being the set-theoretical one. \square

2.3 Definition: (i) For an Ω_p -algebra X , $\alpha \in \Omega_p$, $x := (x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$ one defines

$$\sum_i \alpha_i x_i := \alpha_X(x).$$

(ii) Put $\delta^j := (\delta_i^j)_{i \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, where δ_i^j is the Kronecker-symbol, i.e. $\delta_i^i = 1$ and $\delta_i^j = 0$ for $j \neq i$ ($i, j \in \mathbb{N}$). \square

2.4 Definition (c.f. [8], 2.2): (i) An Ω_p -algebra $(X, (\alpha_X, \alpha \in \Omega_p))$ is called a p-totally convex space if and only if $X \neq \emptyset$ and the following two axioms are satisfied:

(TC_p1) For all $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}, j \in \mathbb{N}, \sum_i \delta_i^j x_i = x_j$ holds.

(TC_p2) For all $\alpha, \beta^i \in \Omega_p$ ($i \in \mathbb{N}$), $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}, \sum_i \alpha_i \left(\sum_j \beta_j^i x_j \right) = \sum_j \left(\sum_i \alpha_i \beta_j^i \right) x_j$ holds.

(ii) The full subcategory of the category of Ω_p -algebras which is determined by all p-totally convex spaces is denoted by **TC_p**. \square

If $D := (X, (\alpha_X, \alpha \in \Omega_p))$ is a p-totally convex space, then we write $x \in D$ instead of $x \in X$ and $D \in \mathbf{TC}_p$ instead of $(X, (\alpha_X, \alpha \in \Omega_p)) \in Ob\mathbf{TC}_p$. \square

2.5 Proposition: In a p-totally convex space D , the following computational rules are valid: (i) Let $\alpha \in \Omega_p$ and let T be a set with $supp\alpha \subset T$; then for all $x = (x_i)_{i \in \mathbb{N}}, y = (y_i)_{i \in \mathbb{N}} \in D^{\mathbb{N}}$ with $x_i = y_i$ ($i \in T$) $\sum_i \alpha_i x_i = \sum_i \alpha_i y_i$.

(ii) Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be a permutation, $\alpha \in \Omega_p$ and $x = (x_i)_{i \in \mathbb{N}} \in D^{\mathbb{N}}$; then $\sum_i \alpha_i x_i = \sum_i \alpha_{\varphi(i)} x_{\varphi(i)}$.

(iii) For $0 = (0)_{i \in \mathbb{N}} \in \Omega_p$ $0_D := \sum_i 0 x_i$ does not depend on $(x_i)_{i \in \mathbb{N}} \in D^{\mathbb{N}}$.

(iv) For all $\alpha \in \Omega_p$ $\sum_i \alpha_i 0_D = 0_D$ holds.

(v) For $x := (x_i)_{i \in \mathbb{N}} \in D^{\mathbb{N}}$ put $suppx := \{i \in \mathbb{N} \mid x_i \neq 0\}$ and let T be a set with $suppx \subset T$. If for $\alpha, \beta \in \Omega_p$, $\alpha_i = \beta_i$ for all $i \in T$, then $\sum_i \alpha_i x_i = \sum_i \beta_i x_i$.

(vi) Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be an injective mapping, $\alpha, \beta \in \Omega_p$ with $\alpha_k = \beta_{\varphi(k)}$ ($k \in \mathbb{N}$) and $\beta_i = 0$ ($i \notin \varphi(\mathbb{N})$). Then for all $x, y \in D^{\mathbb{N}}$ with $x_{\varphi(k)} = y_k$ ($k \in \mathbb{N}$) $\sum_i \beta_i x_i = \sum_i \alpha_i y_i$ holds.

(vii) Let $\varphi : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be a bijective mapping with the components $\varphi_1, \varphi_2 : \mathbb{N} \rightarrow \mathbb{N}$, i.e. $\varphi(n) = (\varphi_1(n), \varphi_2(n))$ ($n \in \mathbb{N}$); then for all $\alpha, \beta^i \in \Omega_p$, $x_j^i \in D$ ($i, j \in \mathbb{N}$) $\sum_i \alpha_i \left(\sum_j \beta_j^i x_j^i \right) = \sum_n \left(\alpha_{\varphi_1(n)} \beta_{\varphi_2(n)}^{\varphi_1(n)} \right) x_{\varphi_2(n)}^{\varphi_1(n)}$.

(viii) For $\alpha, \beta \in \Omega_p$, $x_j^i \in D$ ($i, j \in \mathbb{N}$) $\sum_i \alpha_i (\sum_j \beta_j x_j^i) = \sum_j \beta_j (\sum_i \alpha_i x_j^i)$ holds.

Proof: The proof of (i)-(vii) is similar to [8], 2.4, resp. [11], 7.1. For the proof of (viii) put $\beta^i := \beta$ ($i \in \mathbb{N}$) in (vii). Then we get

$$\begin{aligned} \sum_i \alpha_i \left(\sum_j \beta_j x_j^i \right) &= \sum_n \left(\alpha_{\varphi_1(n)} \beta_{\varphi_2(n)} \right) x_{\varphi_2(n)}^{\varphi_1(n)} \\ &= \sum_n \left(\beta_{\varphi_2(n)} \alpha_{\varphi_1(n)} \right) x_{\varphi_2(n)}^{\varphi_1(n)} = \sum_j \beta_j \left(\sum_i \alpha_i x_j^i \right). \end{aligned}$$

□

2.6 Definition: Put $\Omega_{p,fin} := \{\alpha \in \Omega_p \mid |supp\alpha| < \infty\}$. Replacing Ω_p by $\Omega_{p,fin}$ in 2.2 one gets the category of $\Omega_{p,fin}$ -algebras. An $\Omega_{p,fin}$ -algebra X is called a p-absolutely convex space if and only if $X \neq \emptyset$ holds and in 2.4 (TC_p1) and the restriction of (TC_p2) to $\alpha, \beta^i \in \Omega_{p,fin}$ ($i \in \mathbb{N}$) is valid. \mathbf{AC}_p denotes the full subcategory of the category of $\Omega_{p,fin}$ -algebras which is determined by all p-absolutely convex spaces. □

2.7 Remark: The statements in 2.5 by restriction of Ω_p to $\Omega_{p,fin}$ are also valid for p-absolutely convex spaces. □

2.8 Definition: Let $D \in \mathbf{TC}_p(\mathbf{AC}_p)$, $x = (x_i)_{i \in \mathbb{N}} \in D^{\mathbb{N}}$ and $\beta = (\beta_i)_{i \in \mathbb{N}} \in \Omega_p(\Omega_{p,fin})$ with $supp\beta = \{1\}$. Then one defines $\beta_1 x_1 := \sum_i \beta_i x_i$. □

By 2.8, on $D \in \mathbf{TC}_p(\mathbf{AC}_p)$ a $\bigcirc(\mathbb{K})$ -scalar-multiplication is defined. By 2.5 we get easily the following

2.9 Proposition: Let $D \in \mathbf{TC}_p(\mathbf{AC}_p)$, $\alpha, \beta \in \bigcirc(\mathbb{K})$, $x \in D$, $(\gamma_i)_{i \in \mathbb{N}} \in \Omega_p(\Omega_{p,fin})$ and $(x_i)_{i \in \mathbb{N}} \in D^{\mathbb{N}}$. Then the following statements are valid:

(i) $1x = x$. (ii) $\alpha 0_D = 0_D$. (iii) $\alpha(\beta x) = (\alpha\beta)x$.

(iv) $\sum_i \gamma_i x = (\sum_i \gamma_i)x$. (v) $\sum_i (\alpha\gamma_i)x_i = \alpha(\sum_i \gamma_i x_i)$. □

As a category of equationally defined universal algebras over \mathbf{Set} , $\mathbf{TC}_p(\mathbf{AC}_p)$ is

complete and cocomplete. Let $D_i \in \mathbf{TC}_p(\mathbf{AC}_p)$ ($i \in I$). On the cartesian product $\prod_{i \in I} D_i$ a $\mathbf{TC}_p(\mathbf{AC}_p)$ -structure is defined componentwise. This $\mathbf{TC}_p(\mathbf{AC}_p)$ -space is denoted by $\prod_{i \in I} D_i$ and together with the canonical projections $\pi_j : \prod_{i \in I} D_i \rightarrow D_j$ ($j \in I$) this yields the product $(\prod_{i \in I} D_i, \pi_j : \prod_{i \in I} D_i \rightarrow D_j)_{j \in I}$ of the family $(D_i)_{i \in I}$ in $\mathbf{TC}_p(\mathbf{AC}_p)$.

In particular, for $D \in \mathbf{TC}_p(\mathbf{AC}_p)$ the cartesian power $D^{\mathbb{N}} := \prod_{i \in \mathbb{N}} D$ is a p-totally (p-absolutely) convex space and we have the following

2.10 Proposition: For $D \in \mathbf{TC}_p(\mathbf{AC}_p)$ and $\alpha \in \Omega_p(\Omega_{p,fin})$ the mapping $\alpha_D : D^{\mathbb{N}} \rightarrow D$ is a $\mathbf{TC}_p(\mathbf{AC}_p)$ -morphism.

Proof: Let $\beta \in \Omega_p(\Omega_{p,fin})$ and $(x_k^j)_{k \in \mathbb{N}} \in D^{\mathbb{N}}$ ($j \in \mathbb{N}$). Then by 2.5 (viii)

$$\begin{aligned} \alpha_D \left(\sum_j \beta_j \left((x_k^j)_{k \in \mathbb{N}} \right) \right) &= \alpha_D \left(\left(\sum_j \beta_j x_k^j \right)_{k \in \mathbb{N}} \right) = \sum_i \alpha_i \left(\sum_j \beta_j x_i^j \right) \\ &= \sum_j \beta_j \left(\sum_i \alpha_i x_i^j \right) = \sum_j \beta_j \alpha_D \left((x_k^j)_{k \in \mathbb{N}} \right) \text{ follows.} \end{aligned}$$

□

2.11 Lemma: For $0 < s \leq p$, $\mathbf{TC}_p(\mathbf{AC}_p)$ is a subcategory of $\mathbf{TC}_s(\mathbf{AC}_s)$.

Proof: This is a consequence of $\Omega_s \subset \Omega_p$ ($\Omega_{s,fin} \subset \Omega_{p,fin}$). □

§3 The Comparison Functor and the Tensor Product

For a p-normed vector space V and $\alpha \in \Omega_{p,fin}$ one defines $\alpha_{\mathcal{O}_{p,fin}(V)} : \mathcal{O}_{p,fin}(V)^{\mathbb{N}} \rightarrow \mathcal{O}_{p,fin}(V)$ by $\alpha_{\mathcal{O}_{p,fin}(V)}((x_i)_{i \in \mathbb{N}}) := \sum_i \alpha_i x_i$ ($x_i \in \mathcal{O}_{p,fin}(V)$, $i \in \mathbb{N}$). This equips $\mathcal{O}_{p,fin}(V)$ with the structure of a p-absolutely convex space denoted by $\widehat{\mathcal{O}}_{p,fin}(V)$. Obviously, the restrictions of \mathbf{Vec}_p -morphisms are \mathbf{AC}_p -morphisms of the associated p-absolutely convex spaces, hence we have a functor $\widehat{\mathcal{O}}_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{AC}_p$. Similarly, if B is a p-Banach space and $\alpha \in \Omega_p$, the mapping $\alpha_{\mathcal{O}_p(B)} : \mathcal{O}_p(B)^{\mathbb{N}} \rightarrow \mathcal{O}_p(B)$ is defined by $\alpha_{\mathcal{O}_p(B)}((x_i)_{i \in \mathbb{N}}) := \sum_i \alpha_i x_i$ ($x_i \in \mathcal{O}_p(B)$, $i \in \mathbb{N}$). This yields a p-totally convex space $\widehat{\mathcal{O}}_p(B)$ and hence a functor $\widehat{\mathcal{O}}_p : \mathbf{Ban}_p \rightarrow \mathbf{TC}_p$.

In a category \mathbf{K} of commutative equationally defined universal algebras one knows ([7], p. 179) that the forgetful functor $U : \mathbf{K} \rightarrow \mathbf{Set}$ possesses a left adjoint. This left adjoint of the forgetful functor $U_p : \mathbf{TC}_p \rightarrow \mathbf{Set}$ (resp. $U_{p,fin} : \mathbf{AC}_p \rightarrow \mathbf{Set}$) will now be constructed explicitly.

3.1 Proposition: The forgetful functor $U_p : \mathbf{TC}_p \rightarrow \mathbf{Set}$ (resp. $U_{p,fin} : \mathbf{AC}_p \rightarrow \mathbf{Set}$) has $\widehat{\bigcap}_p \circ l_p$ (resp. $\widehat{\bigcap}_{p,fin} \circ l_{p,fin}$) as a left adjoint.

Proof: Let X be a set, $F := \widehat{\bigcap}_p \circ l_p$ and define the mapping $\eta_X : X \rightarrow U_p \circ F(X)$ by $\eta_X(x) := \delta^x$ ($x \in X$). Let $f : X \rightarrow U_p(D)$ ($D \in \mathbf{TC}_p$) be a set mapping. Because of $g = \sum_{x \in \text{supp } g} g(x)\delta^x$ ($g \in F(X)$) ($\text{supp } g$ is countable and can therefore be imbedded into \mathbb{N}) a \mathbf{TC}_p -morphism $\varphi : F(X) \rightarrow D$ with $U_p(\varphi) \circ \eta_X = f$ is uniquely determined by $\varphi(\delta^x) = f(x)$ ($x \in X$). Taking this as a definition one gets the mapping $\varphi : F(X) \rightarrow D$ with $\varphi(g) := \sum_{x \in \text{supp } g} g(x)f(x)$ ($g \in F(X)$). $\varphi(g)$ is well defined, since the definition of $\varphi(g)$ does not depend on the chosen imbedding of $\text{supp } g$ into \mathbb{N} , (cf. 2.5). Let $\alpha \in \Omega_p$, $g_i \in F(X)$ ($i \in \mathbb{N}$) and let S be a countable set with $\bigcup_{i \in \mathbb{N}} \text{supp } g_i \subset S$ and $S \hookrightarrow \mathbb{N}$ be an imbedding of S into \mathbb{N} . Then

$$\begin{aligned} \varphi\left(\sum_i \alpha_i g_i\right) &= \varphi\left(\sum_i \alpha_i \left(\sum_{x \in S} g_i(x)\delta^x\right)\right) = \varphi\left(\sum_{x \in S} \left(\sum_i \alpha_i g_i(x)\right)\delta^x\right) \\ &= \sum_{x \in S} \left(\sum_i \alpha_i g_i(x)\right) f(x) = \sum_i \alpha_i \left(\sum_{x \in S} g_i(x)f(x)\right) = \sum_i \alpha_i \varphi(g_i) \text{ follows.} \end{aligned}$$

Obviously, the equation $U_p(\varphi) \circ \eta_X = f$ is fulfilled and F is a left adjoint. The proof in the finitary case is analogous. \square

Similar to 1.4, one gets for the counit $\varepsilon : \widehat{\bigcap}_p \circ l_p \circ U_p \rightarrow \mathbf{TC}_p$ (resp. $\varepsilon : \widehat{\bigcap}_{p,fin} \circ l_{p,fin} \circ U_{p,fin} \rightarrow \mathbf{AC}_p$) of the adjunction $(U_p, \widehat{\bigcap}_p \circ l_p, \varepsilon, \eta)$ (resp. $(U_{p,fin}, \widehat{\bigcap}_{p,fin} \circ l_{p,fin}, \varepsilon, \eta)$) $\varepsilon_D(\delta^x) = x$ ($x \in D$), $D \in \mathbf{TC}_p$ (resp. $D \in \mathbf{AC}_p$) and ε_D ($D \in \mathbf{TC}_p$) (resp. $D \in \mathbf{AC}_p$) is uniquely determined by this equations. \square

A well-known result about categories of equationally defined universal algebras over \mathbf{Set} ([7], p.179) implies that the forgetful functor $U_p : \mathbf{TC}_p \rightarrow \mathbf{Set}$ resp. $U_{p,fin} : \mathbf{AC}_p \rightarrow \mathbf{Set}$ is monadic. \square

3.2 Corollary: The forgetful functor $G_p : \mathbf{TC}_p \rightarrow \mathbf{AC}_p$ has a left adjoint.

Proof: By 3.1, the forgetful functors $U_p : \mathbf{TC}_p \rightarrow \mathbf{Set}$ and $U_{p,fin} : \mathbf{AC}_p \rightarrow \mathbf{Set}$ possess a left adjoint. Obviously, $U_p = U_{p,fin} \circ G_p$ holds. The category \mathbf{TC}_p possesses coequalizers and the counit ε of the adjunction $(U_{p,fin}, \widehat{\bigcirc}_{p,fin} \circ l_{p,fin}, \varepsilon, \eta)$ is pointwise a regular epimorphism and our assertion follows. \square

3.3 Theorem: (i) $U_p : \mathbf{TC}_p \rightarrow \mathbf{Set}$ is the Eilenberg-Moore category of $\bigcirc_p : \mathbf{Ban}_p \rightarrow \mathbf{Set}$ with $\widehat{\bigcirc}_p : \mathbf{Ban}_p \rightarrow \mathbf{TC}_p$ the comparison functor. $\widehat{\bigcirc}_p$ is full and faithful.

(ii) $U_{p,fin} : \mathbf{AC}_p \rightarrow \mathbf{Set}$ is the Eilenberg-Moore category of $\bigcirc_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{AC}_p$ with $\widehat{\bigcirc}_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{AC}_p$ the comparison functor. $\widehat{\bigcirc}_{p,fin}$ is full and faithful.

Proof: (i) By 1.4, $l_p : \mathbf{Set} \rightarrow \mathbf{Ban}_p$ is left adjoint to \bigcirc_p with unit $\eta_X : X \rightarrow \bigcirc_p \circ l_p(X)$, $\eta_X(x) = \delta^x$ ($x \in X, X \in \mathbf{Set}$), and counit $\varepsilon_B : l_p \circ \bigcirc_p(B) \rightarrow B$, uniquely determined by $\varepsilon_B(\delta^x) = x$ ($x \in \bigcirc_p(B), B \in \mathbf{Ban}_p$). By 3.1, $\widehat{\bigcirc}_p \circ l_p : \mathbf{Set} \rightarrow \mathbf{TC}_p$ is left adjoint to U_p with unit η_X ($X \in \mathbf{Set}$) and counit $\varepsilon'_D : \widehat{\bigcirc}_p \circ l_p \circ U_p(D) \rightarrow D$, uniquely determined by $\varepsilon'_D(\delta^x) = x$ ($x \in D, D \in \mathbf{TC}_p$). Because of $U_p \circ \widehat{\bigcirc}_p = \bigcirc_p$ a routine calculation shows that the monads of these adjunctions are equal ([10], 2.4). By 1.4 we are finished. The proof of (ii) is similar. \square

3.4 Proposition: For the forgetful functors $U_p : \mathbf{TC}_p \rightarrow \mathbf{Set}$ (resp. $U_{p,fin} : \mathbf{AC}_p \rightarrow \mathbf{Set}$) $U_p \simeq \mathbf{TC}_p(\widehat{\bigcirc}_p(\mathbb{K}), -)$ (resp. $U_{p,fin} \simeq \mathbf{AC}_p(\widehat{\bigcirc}_{p,fin}(\mathbb{K}), -)$) holds, in particular $\widehat{\bigcirc}_p(\mathbb{K})$ (resp. $\widehat{\bigcirc}_{p,fin}(\mathbb{K})$) is a generator in \mathbf{TC}_p (\mathbf{AC}_p).

Proof: For $D \in \mathbf{TC}_p$ define the mapping $\rho_D : U_p(D) \rightarrow \mathbf{TC}_p(\widehat{\bigcirc}_p(\mathbb{K}), D)$ by $\rho_D(x)(\alpha) := \alpha x$ ($x \in U_p(D), \alpha \in \widehat{\bigcirc}_p(\mathbb{K})$). Obviously, this defines a bijective natural transformation. The proof for $U_{p,fin}$ is similar. \square

Let $C, D \in \mathbf{TC}_p$ (\mathbf{AC}_p), $\varphi_i \in \mathbf{TC}_p(C, D)$ ($\mathbf{AC}_p(C, D)$) ($i \in \mathbb{N}$) and $\alpha \in \Omega_p$ ($\Omega_{p,fin}$). Define the mapping $\sum_i \alpha_i \varphi_i : C \rightarrow D$ by $(\sum_i \alpha_i \varphi_i)(x) := \sum_i \alpha_i \varphi_i(x)$ ($x \in C$). Then (cf. 2.10, [6], p. 640) we have the

3.5 Proposition: With the above definition $\mathbf{TC}_p(C, D)$ ($\mathbf{AC}_p(C, D)$) is a p-totally (p-absolutely) convex subspace of $D^{U_p(C)}$ ($D^{U_{p,fin}(C)}$). \square

3.6 Definition: (i) For $C, D \in \mathbf{TC}_p$ the above p-totally convex space with underlying set $\mathbf{TC}_p(C, D)$ is denoted by $\text{Hom}_p(C, D)$.

(ii) For \mathbf{TC}_p -morphisms $g : C' \rightarrow C, h : D \rightarrow D'$ one defines the mapping $\text{Hom}_p(g, h) : \text{Hom}_p(C, D) \rightarrow \text{Hom}_p(C', D')$ by $\text{Hom}_p(g, h)(f) := h \circ f \circ g$ ($f \in \text{Hom}_p(C, D)$). \square

Obviously, $\text{Hom}_p(g, h)$ in 3.6(ii) is a \mathbf{TC}_p -morphism. $\text{Hom}_{p,fin}(C, D)$ for p-absolutely convex spaces C, D and $\text{Hom}_{p,fin}(g, h)$ for \mathbf{AC}_p -morphisms $g : C' \rightarrow C, h : D \rightarrow D'$ is defined as in the infinitary case. Now we have

3.7 Proposition: $\text{Hom}_p(-, -) : \mathbf{TC}_p \times \mathbf{TC}_p \rightarrow \mathbf{TC}_p$ and $\text{Hom}_{p,fin}(-, -) : \mathbf{AC}_p \times \mathbf{AC}_p \rightarrow \mathbf{AC}_p$ are internal Hom -functors (in the sense of [6], p. 637) for \mathbf{TC}_p and \mathbf{AC}_p . \square

From 2.10 and [6], p. 640 one gets the following

3.8 Proposition: \mathbf{TC}_p (resp. \mathbf{AC}_p) is an autonomous category in the sense of Linton, i.e. possesses a tensor product, which, together with coherence morphisms turns it into a symmetric monoidal closed category. The induced functor $- \otimes - : \mathbf{TC}_p \times \mathbf{TC}_p \rightarrow \mathbf{TC}_p$ (resp. $- \otimes - : \mathbf{AC}_p \times \mathbf{AC}_p \rightarrow \mathbf{AC}_p$) is a left adjoint of the internal Hom -functor $\text{Hom}_p(-, -) : \mathbf{TC}_p \times \mathbf{TC}_p \rightarrow \mathbf{TC}_p$ (resp. $\text{Hom}_{p,fin}(-, -) : \mathbf{AC}_p \times \mathbf{AC}_p \rightarrow \mathbf{AC}_p$). \square

§4 Congruence Relations

4.1 Definition: Let $D \in \mathbf{TC}_p$ and “ \sim ” an equivalence relation on D . Then “ \sim ” is called a congruence relation if and only if for all $a_i, b_i \in D$ ($i \in \mathbb{N}$), $\alpha \in \Omega_p$, $a_i \sim b_i$ ($i \in \mathbb{N}$) implies $\sum_i \alpha_i a_i \sim \sum_i \alpha_i b_i$. Congruence relations in \mathbf{AC}_p are defined analogously. \square

Obviously, for $D \in \mathbf{TC}_p$ (\mathbf{AC}_p) and a congruence relation “ \sim ” on D the quotient D/\sim possesses a uniquely structure as p-totally (p-absolutely) convex space such that the canonical projection $\pi : D \rightarrow D/\sim$ is a \mathbf{TC}_p (\mathbf{AC}_p)-morphism. \square

4.2 Definition: Let $D \in \mathbf{AC}_p$. The mapping $\| \|_D : D \rightarrow \mathbb{R}$ is defined as a Minkowski-functional by $\|x\|_D := \inf\{|\lambda| \mid \lambda \in \mathbb{O}(\mathbb{K}) \text{ and } x = \lambda y\}$ ($x \in D$). Often one writes simply $\|x\|$ instead of $\|x\|_D$. \square

4.3 Proposition: Let $D \in \mathbf{AC}_p$. Then one has

- (i) For every $x \in D$ $\|x\| = \inf\{\lambda \in [0, 1] \mid \text{there exists } y \in D \text{ with } x = \lambda y\}$.
- (ii) For each \mathbf{AC}_p -morphism $f : C \rightarrow D$, $\|f(x)\|_D \leq \|x\|_C$, $x \in C$, holds.
- (iii) If “ \sim ” is a congruence relation on D and $\pi : D \rightarrow D/\sim$ the canonical projection, then $\|\pi(y)\|_{D/\sim} = \inf\{\|x\|_D \mid \pi(x) = \pi(y)\}$ ($y \in D$).

Proof: (i) follows from $\lambda y = |\lambda| \left(\frac{\lambda}{|\lambda|} y \right)$, $y \in D$, $\lambda \in \mathbb{O}(\mathbb{K}) \setminus \{0\}$ and (ii) is trivial.

(iii) Let $y \in D$ and define $\gamma := \inf\{\|x\|_D \mid \pi(x) = \pi(y)\}$. From (ii) we get $\|\pi(y)\| \leq \gamma$. Let $\lambda \in [0, 1]$, $x' \in D$ with $\pi(y) = \lambda\pi(x') = \pi(\lambda x')$. This implies $\gamma \leq \lambda$, thus $\gamma \leq \|\pi(y)\|$, and we are finished. \square

4.4 Lemma: Let $D \in \mathbf{TC}_p$ (\mathbf{AC}_p) and $x \in D$. Then $\|x\| = \inf\{\|\alpha\| \mid \alpha \in \Omega_p (\Omega_{p,fin}) \text{ and there are } x_i \in D \text{ } (i \in \mathbb{N}) \text{ with } x = \sum_i \alpha_i x_i\}$ holds.

Proof: Obviously, the assertion holds for $x = 0$. In case $x \neq 0$ let $x_i \in D$ ($i \in \mathbb{N}$), $\alpha \in \Omega_p (\Omega_{p,fin})$ with $x = \sum_i \alpha_i x_i$. This implies $\|\alpha\| > 0$ and $y := \sum_i \frac{\alpha_i}{\|\alpha\|} x_i$ is well defined. From $x = \|\alpha\|y$ the assertion follows. \square

4.5 Proposition (cf. [8], 6.2): For all $D \in \mathbf{TC}_p$ (\mathbf{AC}_p), $\alpha \in \Omega_p (\Omega_{p,fin})$, $x_i \in D$ ($i \in \mathbb{N}$) $\|\sum_i \alpha_i x_i\|^p \leq \sum_i |\alpha_i|^p \|x_i\|^p$ holds.

Proof: Let $\varepsilon > 0$. Then there exist $\lambda_i \in [0, 1]$, $y_i \in D$ with $x_i = \lambda_i y_i$ and $\|x_i\| \leq \lambda_i \leq \min\{\|x_i\| + \varepsilon^{\frac{1}{p}}, 1\}$ ($i \in \mathbb{N}$). 4.4 yields $\|\sum_i \alpha_i x_i\|^p = \|\sum_i \alpha_i (\lambda_i y_i)\|^p = \|\sum_i (\alpha_i \lambda_i) y_i\|^p \leq \sum_i |\alpha_i \lambda_i|^p \leq \sum_i |\alpha_i|^p \left(\|x_i\| + \varepsilon^{\frac{1}{p}} \right)^p \leq \sum_i |\alpha_i|^p (\|x_i\|^p + \varepsilon) = \leq (\sum_i |\alpha_i|^p \|x_i\|^p) + \varepsilon$. \square

4.6 Proposition (cf. [8], 6.5): For a family $(D_i)_{i \in I}$ of p-totally (p-absolutely) convex spaces and $x = (x_i)_{i \in I} \in \prod_{i \in I} D_i$, $\|(x_i)_{i \in I}\| = \sup\{\|x_i\| \mid i \in I\}$ holds.

Proof: Define $\sigma := \sup\{\|x_i\| \mid i \in I\}$. It follows immediately from 4.3(ii) that $\sigma \leq \|(x_i)_{i \in I}\|$. On the other hand let us assume that there exists a λ with $\sigma < \lambda < \|x\|$. Then there are $y_i \in D_i$ with $x_i = \lambda y_i$ ($i \in I$), implying $x = \lambda(y_i)_{i \in I}$. This contradicts $\lambda < \|x\|$ and we get $\|x\| = \sup\{\|x_i\| \mid i \in I\}$. \square

4.7 Definition: For an \mathbf{AC}_p -morphism $f : C \rightarrow D$ one puts $\|f\|_s := \inf\{\lambda \in [0, 1] \mid \|f(x)\| \leq \lambda\|x\| \text{ for all } x \in C\}$. \square

4.8 Proposition: If $f : C \rightarrow D$ is an \mathbf{AC}_p -morphism then the following statements are hold: (i) $\|f(x)\| \leq \|f\|_s\|x\|$ for every $x \in C$.

- (ii) $\|f\|_s = \sup\{\|f(x)\| \mid x \in C\}$.
- (iii) If there exists $x_0 \in C$ with $\|x_0\| \neq 0$, then $\|f\|_s = \sup\left\{\frac{\|f(x)\|}{\|x\|} \mid x \in C \text{ with } \|x\| \neq 0\right\}$.

Proof: (i) This follows from 4.7.

(ii) Put $\sigma := \sup\{\|f(x)\| \mid x \in C\}$. Because of $\|x\| \leq 1$ ($x \in C$) (i) implies $\sigma \leq \|f\|_s$. For all $x, y \in C$, $\lambda \in [0, 1]$, $x = \lambda y$ implies $\|f(x)\| = \|f(\lambda y)\| = \|\lambda f(y)\| \leq \lambda\|f(y)\| \leq \lambda\sigma$. This leads to $\|f(x)\| \leq \sigma\|x\|$ ($x \in C$), thus $\|f\|_s \leq \sigma$ and finally $\|f\|_s = \sigma = \sup\{\|f(x)\| \mid x \in C\}$.

(iii) Because of the assumption, $\sigma := \sup\left\{\frac{\|f(x)\|}{\|x\|} \mid x \in C \text{ with } \|x\| \neq 0\right\}$ is well defined. From (i) we get $\sigma \leq \|f\|_s$, and from the definition of σ $\|f(x)\| \leq \sigma\|x\|$ ($x \in C$ with $\|x\| \neq 0$). This is also true for $x \in C$ with $\|x\| = 0$ (4.3(ii)) and implies $\|f\|_s \leq \sigma$, thus $\|f\|_s = \sup\left\{\frac{\|f(x)\|}{\|x\|} \mid x \in C \text{ with } \|x\| \neq 0\right\}$. \square

4.9 Definition: For $0 < \gamma \leq 1$ one defines $\bigcirc^\gamma(\mathbb{IK}) := \{\beta \in \bigcirc(\mathbb{IK}) \mid |\beta| \leq \gamma\}$ and $\overset{\circ}{\bigcirc}{}^\gamma(\mathbb{IK}) := \{\beta \in \mathbb{IK} \mid |\beta| < \gamma\}$. For $\gamma = 1$ one simply writes $\bigcirc(\mathbb{IK})$ ($\overset{\circ}{\bigcirc}(\mathbb{IK})$) instead of $\bigcirc^1(\mathbb{IK})$ ($\overset{\circ}{\bigcirc}^1(\mathbb{IK})$). \square

In the category \mathbf{AC} (and \mathbf{TC}) the following central theorem is valid: Let $D \in \mathbf{AC}$ and “ \sim ” a congruence relation on D . Then, for $x, y \in D$, $S := \{\alpha \in \bigcirc(\mathbb{IK}) \mid \alpha x \sim$

$\alpha y\}$ equals $\{0\}$, $\overset{\circ}{\bigcirc}(\text{IK})$ or $\bigcirc(\text{IK})$ ([8], 4.1, and [3], 1.1). Furthermore, for $x, y, z \in D$, $M := \{\alpha \in [0, 1] \mid \alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z\}$ equals $\{0\}$, $[0, 1[$ or $[0, 1]$ ([2], 1.5). The following theorem generalizes this result to the case $p < 1$. \square

4.10 Theorem: Let $p < 1$, $D \in \mathbf{AC}_p$ and “ \sim ” a congruence relation on D . Then for all $x, y, z \in D$, $\sigma, \tau, \gamma \in]0, 1]$, $x_0, y_0, z_0 \in D$ with $x = \sigma x_0$, $y = \tau y_0$ and $z = \gamma z_0$, and for every $\alpha \in]0, 1[$ with $\alpha x \sim \alpha y$, one puts

$$S := \{\beta \in [0, 1] \mid \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z \sim \beta^{\frac{1}{p}}y + (1 - \beta)^{\frac{1}{p}}z\} \text{ and has:}$$

$$(i) \text{ For } \gamma < 1 \leq \sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}} \left[0, \frac{1-\gamma^p}{\left(\sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}}\right)^{1-p} - \gamma^p} \right] \subset S.$$

$$(ii) \text{ For } \sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}} < 1 \quad S = [0, 1] \text{ holds, in particular } x \sim y.$$

Proof: Obviously, one has $0 \in S$. For $0 < \beta \leq 1$ and all ε with $0 < \varepsilon < \min\{1, \alpha^{-1}\beta^{\frac{1}{p}}\}$ a mapping $f_{\varepsilon, \beta} : [\varepsilon\alpha, \beta^{\frac{1}{p}}] \rightarrow \mathbb{R}$ is defined by $f_{\varepsilon, \beta}(t) := ((t - \varepsilon\alpha)\sigma)^p + ((\beta^{\frac{1}{p}} - t)\tau)^p + \varepsilon^p + (1 - \beta)\gamma^p$ ($t \in [\varepsilon\alpha, \beta^{\frac{1}{p}}]$). $f_{\varepsilon, \beta}$ is differentiable in $[\varepsilon\alpha, \beta^{\frac{1}{p}}]$ with $f'_{\varepsilon, \beta}(t) = p((t - \varepsilon\alpha)^{p-1}\sigma^p - (\beta^{\frac{1}{p}} - t)^{p-1}\tau^p)$ ($t \in [\varepsilon\alpha, \beta^{\frac{1}{p}}]$). For $t_0^\varepsilon := (\beta^{\frac{1}{p}}\sigma^{\frac{p}{1-p}} + \varepsilon\alpha\tau^{\frac{p}{1-p}})(\sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}})^{-1}$, $\varepsilon\alpha < t_0^\varepsilon < \beta^{\frac{1}{p}}$ follows and an elementary computation shows $f'_{\varepsilon, \beta}(t) > 0$ if and only if $t < t_0^\varepsilon$, and $f'_{\varepsilon, \beta}(t) < 0$ if and only if $t > t_0^\varepsilon$ ($\varepsilon\alpha < t < \beta^{\frac{1}{p}}$). Since $f_{\varepsilon, \beta}$ is continuous in $\varepsilon\alpha$ and $\beta^{\frac{1}{p}}$, $f_{\varepsilon, \beta}$ has an absolute maximum in t_0^ε . This leads to $\lim_{\varepsilon \rightarrow 0} f_{\varepsilon, \beta}(t_0^\varepsilon) = \beta \left(\sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}} \right)^{1-p} + (1 - \beta)\gamma^p$. Hence $\lim_{\varepsilon \rightarrow 0} f_{\varepsilon, \beta}(t_0^\varepsilon) < 1$ is equivalent with $\beta \left(\left(\sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}} \right)^{1-p} - \gamma^p \right) < 1 - \gamma^p$. Consequently, in (i) $\lim_{\varepsilon \rightarrow 0} f_{\varepsilon, \beta}(t_0^\varepsilon) < 1$ is equivalent with

$$\beta < (1 - \gamma^p) \left(\left(\sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}} \right)^{1-p} - \gamma^p \right)^{-1}, \text{ and using the assumption in (ii)} \\ \lim_{\varepsilon \rightarrow 0} f_{\varepsilon, \beta}(t_0^\varepsilon) < 1 \text{ is fulfilled for all } \beta \in]0, 1].$$

Hence, for every $\beta \in]0, 1]$ with $\lim_{\varepsilon \rightarrow 0} f_{\varepsilon, \beta}(t_0^\varepsilon) < 1$ there exists an ε_0 with $0 < \varepsilon_0 < \min\{1, \alpha^{-1}\beta^{\frac{1}{p}}\}$, such that for all $\varepsilon \in]0, \varepsilon_0]$, $t \in [\varepsilon\alpha, \beta^{\frac{1}{p}}]$, $f_{\varepsilon, \beta}(t) \leq f_{\varepsilon, \beta}(t_0^\varepsilon) < 1$ holds. Put $M := \{\lambda \in [0, \beta^{\frac{1}{p}}] \mid (\beta^{\frac{1}{p}}\sigma)x_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \sim (\lambda\sigma)x_0 + ((\beta^{\frac{1}{p}} - \lambda)\tau)y_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0\}$. M is well-defined, since $(\beta^{\frac{1}{p}}\sigma)^p + ((1 - \beta)^{\frac{1}{p}}\gamma)^p$

$= \beta\sigma^p + (1 - \beta)\gamma^p \leq 1$ (and $(\beta^{\frac{1}{p}}\tau)^p + ((1 - \beta)^{\frac{1}{p}}\gamma)^p \leq 1$) holds, and for all $\lambda \in [0, \beta^{\frac{1}{p}}]$, $(\lambda\sigma)^p + ((\beta^{\frac{1}{p}} - \lambda)\tau)^p + (1 - \beta)\gamma^p \leq 1$. This can be seen as follows: For $\lambda = \beta^{\frac{1}{p}}$ it was shown above. For $\lambda \in M$ with $\lambda < \beta^{\frac{1}{p}}$ there exists an $\varepsilon \in]0, \varepsilon_0]$ with $t := \lambda + \varepsilon\alpha \leq \beta^{\frac{1}{p}}$. For $\lambda = t - \varepsilon\alpha$, $t \in [\varepsilon\alpha, \beta^{\frac{1}{p}}]$, we get $(\lambda\sigma)^p + ((\beta^{\frac{1}{p}} - \lambda)\tau)^p + (1 - \beta)\gamma^p = ((t - \varepsilon\alpha)\sigma)^p + ((\beta^{\frac{1}{p}} - t)\tau + \varepsilon\alpha\tau)^p + (1 - \beta)\gamma^p \leq ((t - \varepsilon\alpha)\sigma)^p + ((\beta^{\frac{1}{p}} - t)\tau)^p + \varepsilon^p + (1 - \beta)\gamma^p = f_{\varepsilon, \beta}(t) \leq f_{\varepsilon, \beta}(t_0^\varepsilon) < 1$. For every $\lambda \in M \setminus \{0\}$ there exists an $\varepsilon_1 \in]0, \varepsilon_0]$ with $\varepsilon_1\alpha \leq \lambda$. For all $\varepsilon \in]0, \varepsilon_1]$ $\mu := \lambda - \varepsilon\alpha \in [0, \lambda[$ holds. If $\lambda \in M \setminus \{0\}$ and $\varepsilon_1, \varepsilon, \mu$ are as above, the following terms are well-defined because M is well-defined. Then we have

$$\begin{aligned} & (\mu\sigma)x_0 + ((\beta^{\frac{1}{p}} - \mu)\tau)y_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \\ &= (\mu\sigma)x_0 + ((\beta^{\frac{1}{p}} - \lambda)\tau + \varepsilon\alpha\tau)y_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \\ &= (\mu\sigma)x_0 + ((\beta^{\frac{1}{p}} - \lambda)\tau)y_0 + \varepsilon(\alpha(\tau y_0)) + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \\ &\sim (\mu\sigma)x_0 + ((\beta^{\frac{1}{p}} - \lambda)\tau)y_0 + \varepsilon(\alpha(\sigma x_0)) + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \\ &= (\mu\sigma)x_0 + ((\beta^{\frac{1}{p}} - \lambda)\tau)y_0 + (\varepsilon\alpha\sigma)x_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \\ &= (\lambda\sigma)x_0 + ((\beta^{\frac{1}{p}} - \lambda)\tau)y_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \end{aligned}$$

$\sim ((\beta^{\frac{1}{p}}\sigma)x_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0$. This implies $\mu \in M$, thus $[\lambda - \varepsilon_1\alpha, \lambda] \subset M$. Obviously, $\beta^{\frac{1}{p}} \in M \setminus \{0\}$ holds. Thus there exists an $\varepsilon_1 \in]0, \varepsilon_0]$ with $\beta^{\frac{1}{p}} - \varepsilon_1\alpha > 0$. By the above equations, this implies $\beta^{\frac{1}{p}} - n_0\varepsilon_1\alpha \in M$, where $n_0 := \max\{n \in \mathbb{N} \mid \beta^{\frac{1}{p}} - n\varepsilon_1\alpha > 0\}$. Because of $\beta^{\frac{1}{p}} - (n_0+1)\varepsilon_1\alpha \leq 0$ there exists an $\varepsilon_2 \in]0, \varepsilon_1]$ with $(\beta^{\frac{1}{p}} - n_0\varepsilon_1\alpha) - \varepsilon_2\alpha = 0$. Again the above equations yield $0 = (\beta^{\frac{1}{p}} - n_0\varepsilon_1\alpha) - \varepsilon_2\alpha \in M$ because of $\beta^{\frac{1}{p}} - n_0\varepsilon_1\alpha \in M \setminus \{0\}$. One has $\beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z = (\beta^{\frac{1}{p}}\sigma)x_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 \sim (\beta^{\frac{1}{p}}\tau)y_0 + ((1 - \beta)^{\frac{1}{p}}\gamma)z_0 = \beta^{\frac{1}{p}}y + (1 - \beta)^{\frac{1}{p}}z$, i.e. $\beta \in S$, and we are finished. \square

4.11 Corollary: If $p < 1$, $D \in \mathbf{AC}_p$ and “ \sim ” is a congruence relation on D , then for $x, y, z \in D$ and $0 < \alpha < 1$ with $\alpha x \sim \alpha y$ one defines

$$S := \{\beta \in [0, 1] \mid \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z \sim \beta^{\frac{1}{p}}y + (1 - \beta)^{\frac{1}{p}}z\}. \text{ Then:}$$

(i) For $\|z\| < 1 \leq \|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}}$

$$\left[0, \frac{1 - \|z\|^p}{\left(\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} \right)^{1-p} - \|z\|^p} \right] \subset S \text{ holds.}$$

(ii) From $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} < 1$, $S = [0, 1]$ follows, in particular $x \sim y$.

Proof: (i) There exist $\sigma_n, \tau_n, \gamma_n \in]0, 1]$, $x_n, y_n, z_n \in D$ with $x = \sigma_n x_n$, $y = \tau_n y_n$, $z = \gamma_n z_n$, $1 \leq \sigma_n^{\frac{p}{1-p}} + \tau_n^{\frac{p}{1-p}}$, $\gamma_n < 1$ ($n \in \mathbb{N}$) and $\lim_{n \rightarrow \infty} \sigma_n = \|x\|$, $\lim_{n \rightarrow \infty} \tau_n = \|y\|$ and $\lim_{n \rightarrow \infty} \gamma_n = \|z\|$. Because of 4.10(i), for all $n \in \mathbb{N}$

$$\left[0, \frac{1 - \gamma_n^p}{\left(\sigma_n^{\frac{p}{1-p}} + \tau_n^{\frac{p}{1-p}} \right)^{1-p} - \gamma_n^p} \right] \subset S \text{ holds and the assertion follows.}$$

(ii) There exist $\sigma, \tau, \gamma \in]0, 1]$, $x_0, y_0, z_0 \in D$ with $x = \sigma x_0$, $y = \tau y_0$, $z = \gamma z_0$ and $\sigma^{\frac{p}{1-p}} + \tau^{\frac{p}{1-p}} < 1$, and the assertion is implied by 4.10(ii). \square

In the continuation of this paper it will be shown that the result obtained in 4.11(i) for the set S is the best possible one; furthermore it will be proved that in the remaining cases i.e. $\|z\| = 1 \leq \|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}}$ “no further statement” about the set S can be made. \square

4.12 Corollary: If “ \sim ” is a congruence relation on D , $D \in \mathbf{AC}_p$, $p < 1$, and if one defines $S := \{\alpha \in \bigcirc(\mathbb{K}) \mid \alpha x \sim \alpha y\}$ for $x, y \in D$, then $S = \{0\}$ or the following statements are fulfilled:

(i) $1 \leq \|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}}$ implies $\overset{\circ}{\bigcirc} \beta_p(\mathbb{K}) \subset S$ for $\beta_p := \left(\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} \right)^{\frac{p-1}{p}}$.

(ii) $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} < 1$ implies $S = \bigcirc(\mathbb{K})$, in particular $x \sim y$.

Proof: (i) Put $z := 0$ in 4.11 and notice that $\alpha x \sim \alpha y$ ($\alpha \in \bigcirc(\mathbb{K})$) implies $\beta x \sim \beta y$ for all $\beta \in \bigcirc(\mathbb{K})$ with $|\beta| = \alpha$. \square

4.13 Corollary: If $D \in \mathbf{AC}_p$, $p < 1$ and S are as in 4.12, then $S = \{0\}$ or $\overset{\circ}{\bigcirc}^{\omega_p}(\mathbb{IK}) \subset S$ (where $\omega_p = (\frac{1}{2})^{\frac{1}{p}-1}$).

Proof: Put $\sigma := 1$, $\tau := 1$, $\gamma := 0$, $x_0 := x$, $y_0 := y$, $z := 0$, $z_0 := 0$ in 4.10. In the same way as in 4.12 one gets the assertion. \square

4.14 Corollary: If “ \sim ” is a congruence relation on $D \in \mathbf{AC}_p$, then for $x, y \in D$ with $\|x\|, \|y\| < \omega_p$, $\alpha x \sim \alpha y$ ($\alpha \in \overset{\circ}{\bigcirc}(\mathbb{IK}) \setminus \{0\}$) implies $x \sim y$.

Proof: There exist $\sigma \in]0, \omega_p[$, $x_0, y_0 \in D$ with $x = \sigma x_0$ and $y = \sigma y_0$. Put $S := \{\beta \in \overset{\circ}{\bigcirc}(\mathbb{IK}) \mid \beta x_0 \sim \beta y_0\}$. $\alpha \sigma \in S$ implies $S \neq \{0\}$. By 4.13, resp. [8], 4.1, $\sigma \in S$ holds, thus $x = \sigma x_0 \sim \sigma y_0 = y$. \square

4.15 Theorem (cf. [8], 6.9): For $x \in D$, $D \in \mathbf{TC}_p$, $\|x\| = 0$ implies $x = 0$.

Proof: For $p = 1$ the assertion is proved in [8], 6.9. Let $p < 1$. If $\|x\| = 0$, define $\varepsilon_n := (\frac{1}{2})^{\frac{n}{p}}$ ($n \in \mathbb{N}_0$). Then $\varepsilon_n \in]0, 1]$ ($n \in \mathbb{N}_0$), $\sum_{n=1}^{\infty} \varepsilon_{n+1}^p \leq \sum_{n=1}^{\infty} \varepsilon_n^p = 1$, and $\varepsilon_2^p + \sum_{n=1}^{\infty} \varepsilon_{n+1}^p \leq 1$ follow. Because of $\|x\| = 0$ there exist $y_n \in D$ with $x = \varepsilon_n y_n$, in particular $\varepsilon_n y_n = \varepsilon_{n+1} y_{n+1} = \varepsilon_n \left(\left(\frac{1}{2} \right)^{\frac{1}{p}} y_{n+1} \right)$ ($n \in \mathbb{N}_0$). For $\alpha := \frac{\omega_p}{2}$ we get $\alpha y_n = \alpha \left(\left(\frac{1}{2} \right)^{\frac{1}{p}} y_{n+1} \right) = \left(\frac{1}{2} \right)^{\frac{1}{p}} (\alpha y_{n+1})$ ($n \in \mathbb{N}_0$). Then for $z := \sum_{n=1}^{\infty} \varepsilon_{n+1} y_{n-1}$, one has

$$\begin{aligned} \alpha z &= \alpha(\varepsilon_2 y_0) + \alpha \left(\sum_{n=1}^{\infty} \varepsilon_{n+2} y_n \right) = \varepsilon_2(\alpha x) + \sum_{n=1}^{\infty} \varepsilon_{n+2}(\alpha y_n) \\ &= \varepsilon_2(\alpha x) + \sum_{n=1}^{\infty} \varepsilon_{n+1} \left(\left(\frac{1}{2} \right)^{\frac{1}{p}} (\alpha y_n) \right) = \varepsilon_2(\alpha x) + \sum_{n=1}^{\infty} \varepsilon_{n+1}(\alpha y_{n-1}) \\ &= \alpha(\varepsilon_2 x) + \alpha \left(\sum_{n=1}^{\infty} \varepsilon_{n+1} y_{n-1} \right) = \alpha(\varepsilon_2 x) + \alpha z. \end{aligned}$$

This implies $0 = \alpha(\alpha(\varepsilon_2 x) + \alpha z) - \alpha(\alpha z) = \alpha^2 \varepsilon_2 x = \alpha \varepsilon_2(\alpha y_0) = \alpha \varepsilon_2 \left(\alpha \left(\frac{1}{2} \right)^{\frac{1}{p}} y_1 \right) = \left(\alpha^2 \varepsilon_2 \left(\frac{1}{2} \right)^{\frac{1}{p}} \right) y_1$. Because of $\varepsilon_1 = \left(\frac{1}{2} \right)^{\frac{1}{p}} < \omega_p$ 4.13 yields $x = \varepsilon_1 y_1 = 0$. \square

4.16 Remark: In order to see that 4.15 is false for $D \in \mathbf{AC}_p$, take an arbitrary

\mathbb{K} -vector space $V \neq \{0\}$. V has a canonical structure of a p -absolutely convex space and one has $\|x\| = 0$ for any $x \in V$. \square

4.17 Theorem: For $D \in \mathbf{AC}_p$ and $x \in D$ the following statements hold:

- (i) $S := \{\alpha \in \bigcirc(\mathbb{K}) \mid \|\alpha x\| = 0\}$ equals $\{0\}$, $\overset{\circ}{\bigcirc}(\mathbb{K})$ or $\bigcirc(\mathbb{K})$. In case $\|x\| < 1$ $S \neq \overset{\circ}{\bigcirc}(\mathbb{K})$.
- (ii) $M := \{\beta \in \bigcirc(\mathbb{K}) \mid \beta x = 0\}$ equals $\{0\}$, $\overset{\circ}{\bigcirc}(\mathbb{K})$ or $\bigcirc(\mathbb{K})$. In case $\|x\| < 1$ $M \neq \overset{\circ}{\bigcirc}(\mathbb{K})$.

Proof: For $p = 1$ the assertions in (i) and (ii) are proved in [8], 4.1, 6.10 resp. are a direct consequence of this.

- (i) For $p < 1$ assume $S \neq \{0\}$. For $\alpha \in S \setminus \{0\}$ and $\beta \in \overset{\circ}{\bigcirc}(\mathbb{K})$ there exists an $\varepsilon_0 \in]0, 1[$ with $|\beta| < \left(1 + \varepsilon_0^{\frac{p}{1-p}}\right)^{\frac{p-1}{p}}$. Since $\|\alpha x\| = 0$, for $\varepsilon \in]0, \varepsilon_0]$ there exists a $y_\varepsilon \in D$ with $\alpha x = (\alpha\varepsilon)y_\varepsilon = \alpha(\varepsilon y_\varepsilon)$, hence $|\alpha|x = |\alpha|(\varepsilon y_\varepsilon)$. Putting $z := 0$, $z_0 := 0$, $\gamma := 0$, $\sigma := 1$, $x_0 := x$, $y := \varepsilon y_\varepsilon$, $y_0 := y_\varepsilon$, $\tau := \varepsilon$, in 4.10, one gets $|\beta|x = |\beta|(\varepsilon y_\varepsilon)$ because of $|\alpha| \in S \setminus \{0\}$ and $|\beta| < \left(1 + \varepsilon_0^{\frac{p}{1-p}}\right)^{\frac{p-1}{p}} \leq \left(1 + \varepsilon^{\frac{p}{1-p}}\right)^{\frac{p-1}{p}}$ from 4.10 (i) or (ii). This leads to $\beta x = \beta(\varepsilon y_\varepsilon) = \varepsilon(\beta y_\varepsilon)$ ($\varepsilon \in]0, \varepsilon_0]$), thus $\|\beta x\| = 0$ i.e. $\overset{\circ}{\bigcirc}(\mathbb{K}) \subset S$. If there exists a $\gamma \in S$ with $|\gamma| = 1$, one gets easily $S = \bigcirc(\mathbb{K})$. Thus S is $\overset{\circ}{\bigcirc}(\mathbb{K})$ or $\bigcirc(\mathbb{K})$.

In case $\|x\| < 1$ there exists a $\sigma \in]0, 1[$, $x_0 \in D$ with $x = \sigma x_0$. Put $S' := \{\tau \in \bigcirc(\mathbb{K}) \mid \|\tau x_0\| = 0\}$. Because of $\|(\alpha\sigma)x_0\| = \|\alpha x\| = 0$ and $\alpha\sigma \neq 0$ we have $S' \neq \{0\}$. This implies $\overset{\circ}{\bigcirc}(\mathbb{K}) \subset S'$, i.e. $\sigma \in S'$ and leads to $\|x\| = \|\sigma x_0\| = 0$ resp. $1 \in S$, implying $S = \bigcirc(\mathbb{K})$.

- (ii) For $p < 1$ assume $M \neq \{0\}$. Put $y := 0$ in 4.12. In case $\|x\| = 1$ 4.12 (i) yields $\overset{\circ}{\bigcirc}(\mathbb{K}) \subset M$, and $M = \bigcirc(\mathbb{K})$ otherwise (4.12 (ii)). If there exists a $\gamma \in M$ with $|\gamma| = 1$, in case $\|x\| = 1$ $M = \bigcirc(\mathbb{K})$ follows immediately. Thus $M = \overset{\circ}{\bigcirc}(\mathbb{K})$ or $\bigcirc(\mathbb{K})$ holds. \square

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p-Banach Spaces and p-Totally Convex Spaces II

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Introduction

In [7] and [8] Pumplün and Röhrl introduced the category \mathbf{TC} (resp. \mathbf{TC}_{fin}) of (finitely) totally convex (t.c.) spaces, which are the Eilenberg-Moore algebras of the monad induced by the unit ball functor from the category of Banach spaces (resp. normed vector spaces) with linear contractions to the category of sets. In accordance with [1] we use the term “absolutely convex”(a.c.) for the spaces Pumplün and Röhrl call “finitely totally convex”; \mathbf{AC} denotes the category of absolutely convex spaces.

In [4] the categories \mathbf{TC}_p of p -totally convex and \mathbf{AC}_p of p -absolutely convex spaces are introduced and investigated ($0 < p < 1$). The notion of p -absolutely convex spaces is a generalization of p -absolutely convex subsets of \mathbb{K} -vector spaces and p -totally convex spaces are a generalization of p -totally convex subsets of topological \mathbb{K} -vector spaces, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

For $p = 1$, \mathbf{TC}_p and \mathbf{AC}_p coincide with the categories \mathbf{TC} and \mathbf{AC} and the results proved by Pumplün and Röhrl in [7], §1-§3, are contained in [4]. Furthermore, a generalization of the central Theorem 4.1 in [7] for p -absolutely (resp. p -totally) convex spaces, $p < 1$, is proved in [4], 4.11.

In the present paper it is shown that this generalization i.e. the results 4.11–4.14 in [4] are the best possible in case $p < 1$. Furthermore, the results proved by Pumplün and Röhrl in [7], §4 and §5, are generalized to $p < 1$, especially the left adjoints S_p and $S_{p,fin}$ of the comparison functors $\widehat{\bigcirc}_p : \mathbf{Ban}_p \rightarrow \mathbf{TC}_p$ and $\widehat{\bigcirc}_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{AC}_p$ are explicitly constructed. \square

For a p -absolutely convex space D a so-called “ p -norm” $\| \| : D \rightarrow \mathbb{R}$ is defined by the Minkowski-functional $\|x\| := \inf\{|\lambda| \mid \lambda \in \bigcirc(\mathbb{K})$ and there is $y \in D$ with $x = \lambda y\}$ ([4], 4.2). Let “ \sim ” be a congruence relation on $D \in \mathbf{TC}_p$ (resp. $D \in \mathbf{AC}_p$), $x, y, z, x_0, y_0, z_0 \in D$, $\sigma, \tau, \gamma \in]0, 1]$ with $x = \sigma x_0, y = \tau y_0$ and $z = \gamma z_0$. If there is an $\alpha \in]0, 1[$ with $\alpha x \sim \alpha y$, then for

$$S := \{\beta \in [0, 1] \mid \beta^{\frac{1}{p}} x + (1 - \beta)^{\frac{1}{p}} z \sim \beta^{\frac{1}{p}} y + (1 - \beta)^{\frac{1}{p}} z\}$$

in case $p < 1$ we have the following central result ([4], 4.11):

(i) If $\|z\| < 1 \leq \|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}}$, then

$$\left[0, \frac{1 - \|z\|^p}{\left(\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} \right)^{1-p} - \|z\|^p} \right] \subset S.$$

(ii) If $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} < 1$, then $S = [0, 1]$.

In 4.22 it will be shown that [4], 4.11(i) is the best possible result. The same holds for [4], 4.12(i), 4.13 and 4.14. For an absolutely convex space D the set $S := \{\beta \in [0, 1] | \beta x + (1 - \beta)z \sim \beta y + (1 - \beta)z\}$ ([1], 1.5) resp. $S := \{\beta \in \mathbb{K} | \beta x \sim \beta y\}$ ([7], 4.1) does not depend on the (1-)norm of the elements. Contrary to this the p-norm plays an important role for $p < 1$. One gets results for elements with “small” p-norm (4.14) that correspond to the absolutely convex case ([2], 1.3, 1.5). If the p-norms of $x, y \in D$ are “big” and $\|z\| = 1$ ($z \in D$), it will be shown in 4.28 that $S := \{\beta \in [0, 1] | \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z \sim \beta^{\frac{1}{p}}y + (1 - \beta)^{\frac{1}{p}}z\}$ is, for $p < 1$, neither convex nor p-convex ([5], p.101). The following chapter about congruence relations is the continuation of [4], §4. \square

§4 Congruence Relations

4.18 Definition: Let $\mathbb{K} := \mathbb{R}, p < 1$, and e_1, e_2, e_3 be the unit-vectors in \mathbb{R}^3 . Define $A := \widehat{\bigcirc}_p \circ l_p(\{e_1, e_2, e_3\}) = \{(x, y, z) \in \mathbb{R}^3 | |x|^p + |y|^p + |z|^p \leq 1\}$. For $\rho \in \mathbb{R}$ a relation “ \sim_ρ ” on the p-totally convex space A is defined in the following way: For $(x_i, y_i, z_i) \in A$ ($i = 1, 2$) $(x_1, y_1, z_1) \sim_\rho (x_2, y_2, z_2)$ if and only if $(x_1, y_1, z_1) = (x_2, y_2, z_2)$ or the following two conditions are satisfied:

(i) $z_1 = z_2$ and $y_1 + \rho x_1 = y_2 + \rho x_2$.

(ii) For all $\lambda \in [0, 1]$

$$|\lambda x_1 + (1 - \lambda)x_2|^p + |\lambda y_1 + (1 - \lambda)y_2|^p + |\lambda z_1 + (1 - \lambda)z_2|^p < 1. \quad \square$$

4.19 Proposition: The above relation " \sim_ρ " ($\rho \in \mathbb{R}$) is a congruence relation on A , i.e. $A_\rho := A/\sim_\rho$ is a p-totally convex space.

Proof: Put " \sim " := " \sim_ρ " for the sake of brevity. Obviously, " \sim " is reflexive and symmetric. Consider $(x_i, y_i, z_i) \in A$ ($i = 1, 2, 3$) with $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ and $(x_2, y_2, z_2) \sim (x_3, y_3, z_3)$. Without loss of generality one may assume $(x_1, y_1, z_1) \neq (x_2, y_2, z_2) \neq (x_3, y_3, z_3)$. (i) is trivially fulfilled. Next, for $\lambda \in [0, 1]$, there exist $n, m \in \mathbb{N}$ with $(n, m) = (1, 2)$ or $(n, m) = (2, 3)$ and $\lambda x_1 + (1 - \lambda)x_3 = \alpha x_n + (1 - \alpha)x_m$ for some $\alpha \in [0, 1]$. Because of (i) there is a $c \in \mathbb{R}$ with $y_i = -\rho x_i + c$ ($i = 1, 2, 3$), which implies $\lambda y_1 + (1 - \lambda)y_3 = \lambda(-\rho x_1 + c) + (1 - \lambda)(-\rho x_3 + c) = -\rho(\lambda x_1 + (1 - \lambda)x_3) + c = -\rho(\alpha x_n + (1 - \alpha)x_m) + c = \alpha(-\rho x_n + c) + (1 - \alpha)(-\rho x_m + c) = \alpha y_n + (1 - \alpha)y_m$. This leads to $|\lambda x_1 + (1 - \lambda)x_3|^p + |\lambda y_1 + (1 - \lambda)y_3|^p + |\lambda z_1 + (1 - \lambda)z_3|^p = |\alpha x_n + (1 - \alpha)x_m|^p + |\alpha y_n + (1 - \alpha)y_m|^p + |\alpha z_n + (1 - \alpha)z_m|^p < 1$, thus " \sim " is transitive. Let $\alpha \in \Omega_p$, $(x_1^i, y_1^i, z_1^i), (x_2^i, y_2^i, z_2^i) \in A$ with $(x_1^i, y_1^i, z_1^i) \sim (x_2^i, y_2^i, z_2^i)$ ($i \in \mathbb{N}$). We may assume $(x_1^{i_0}, y_1^{i_0}, z_1^{i_0}) \neq (x_2^{i_0}, y_2^{i_0}, z_2^{i_0})$ for some $i_0 \in \text{supp } \alpha$. For $\lambda \in [0, 1]$, (ii) yields

$$\begin{aligned} & |\lambda \sum_i \alpha_i x_1^i + (1 - \lambda) \sum_i \alpha_i x_2^i|^p + |\lambda \sum_i \alpha_i y_1^i + (1 - \lambda) \sum_i \alpha_i y_2^i|^p \\ & + |\lambda \sum_i \alpha_i z_1^i + (1 - \lambda) \sum_i \alpha_i z_2^i|^p \\ & \leq \sum_i |\alpha_i|^p (|\lambda x_1^i + (1 - \lambda)x_2^i|^p + |\lambda y_1^i + (1 - \lambda)y_2^i|^p + |\lambda z_1^i + (1 - \lambda)z_2^i|^p) \\ & < \sum_i |\alpha_i|^p \leq 1. \text{ Hence } "\sim" \text{ is a congruence relation on } A. \quad \square \end{aligned}$$

4.20 Definition: If $\pi_\rho : A \rightarrow A_\rho$ ($\rho \in \mathbb{R}$) is the canonical projection one simply writes $\overline{(x, y, z)} := \pi_\rho((x, y, z))$, if no misunderstandings are possible. \square

4.21 Lemma: If $p < 1$, then, for all elements $(x, y, 0) \in A$ with $|x|^p + |y|^p < 1$, there exist $a, b \in \bigcirc(\mathbb{R})$ with $ab = 0$ and $(x, y, 0) \sim (a, b, 0)$.

Proof: Denote " \sim_γ " by " \sim " and define the mappings $g, h : [0, 1] \rightarrow \mathbb{R}$ by $g(\lambda) := \lambda^p|x|^p + |y + (1 - \lambda)x|^p$ and $h(\lambda) := \lambda^p|y|^p + |x + (1 - \lambda)y|^p$ ($\lambda \in [0, 1]$). In case $x = 0$ or $y = 0$ there is nothing to prove. So assume $xy \neq 0$. By 4.18, $(x, y, 0) \sim (0, x + y, 0)$ is equivalent with $g(\lambda) < 1$ for all $\lambda \in [0, 1]$ and $(x, y, 0) \sim (x + y, 0, 0)$ is equivalent with $h(\lambda) < 1$ for all $\lambda \in [0, 1]$.

First, let $0 < x \leq y$. g is differentiable in $]0, 1[$ with $g'(\lambda) = p(\lambda^{p-1}x^p - (y + (1 - \lambda)x)^{p-1}x)$. $g'(\lambda) = 0$ is equivalent with $\lambda = \frac{x+y}{2x}$. From $\frac{x+y}{2x} \geq 1$, $g(0) = (x+y)^p \leq x^p + y^p < 1$ and $g(1) = x^p + y^p < 1$ one concludes $g(\lambda) < 1$ for $\lambda \in [0, 1]$, hence $(x, y, 0) \sim (0, x+y, 0)$. If $0 < y \leq x$ one gets $h(\lambda) < 1$ ($\lambda \in [0, 1]$), thus $(x, y, 0) \sim (x+y, 0, 0)$ by a symmetry argument.

Consider now $0 < x$ and $y < 0$. In case $x+y \leq 0$, for $\lambda \in [0, 1]$, $y + (1 - \lambda)x \leq 0$ and $g'(\lambda) = p(\lambda^{p-1}x^p + (-y + (1 - \lambda)x))^{p-1}x) > 0$ holds. Because of $g(1) = x^p + (-y)^p < 1$ and $g(0) < 1$, $g(\lambda) < 1$ ($\lambda \in [0, 1]$) follows, hence $(x, y, 0) \sim (0, x+y, 0)$. If $x+y \geq 0$, $0 < -y, -x < 0$ and $(-y) + (-x) \leq 0$ holds, and by symmetry one gets $h(\lambda) < 1$ ($\lambda \in [0, 1]$), thus $(x, y, 0) \sim (x+y, 0, 0)$.

If $x < 0 < y$ and $x+y \geq 0$, $(x, y, 0) = -(-x, -y, 0) \sim -(0, -(x+y), 0) = (0, x+y, 0)$ follows. Analogously, $x+y \leq 0$ yields $(x, y, 0) = -(-x, -y, 0) \sim -(-(x+y), 0, 0) = (x+y, 0, 0)$. The assertion in the last case, $x < 0, y < 0$, follows immediately from the above considerations. \square

4.22 Theorem: For $p < 1$, $0 < \gamma < 1$, $1 \leq r \leq 2$, $\mathbb{IK} := \mathbb{IR}$ define $\sigma := (\frac{r}{2})^{\frac{1-p}{p}}$, $x := \sigma \overline{e_1}$, $y := \sigma \overline{e_2}$, $z := \gamma \overline{e_3} \in A_1$. Then $\|x\| = \|y\| = \sigma$, $\|z\| = \gamma$, $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = r$ and for

$$S := \{\beta \in [0, 1] | \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z = \beta^{\frac{1}{p}}y + (1 - \beta)^{\frac{1}{p}}z\}$$

$$S = \left[0, \frac{1 - \|z\|^p}{(\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}})^{1-p} - \|z\|^p} \right] \text{ holds.}$$

Proof: Put “ \sim ”:=“ $\tilde{\sim}$ ” (4.18, 4.19). Obviously one has $\|z\| \leq \gamma$. Assume that there exists an ϵ with $0 < \epsilon < \gamma$ and $z = \epsilon \overline{(a, b, c)}$ ($\overline{(a, b, c)} \in A_1$). Then $\gamma = \epsilon c = \epsilon |c| \leq \epsilon$ follows (4.18), contradicting $\epsilon < \gamma$, i.e. $\|z\| = \gamma$. As $\|x\| \leq \sigma$, assume that there exists an ϵ with $0 < \epsilon < \sigma$ and $x = \epsilon \overline{(a, b, c)}$. Then $c = 0$ because of 4.18. For any $\epsilon' \in]\epsilon, \sigma[$, $x = \epsilon \overline{(a, b, c)} = \epsilon' \overline{(\frac{\epsilon}{\epsilon'} a, \frac{\epsilon}{\epsilon'} b, 0)}$ and $\|\overline{(\frac{\epsilon}{\epsilon'} a, \frac{\epsilon}{\epsilon'} b, 0)}\| = \|\overline{\frac{\epsilon}{\epsilon'}(a, b, 0)}\| \leq \frac{\epsilon}{\epsilon'} < 1$ follows.

By 4.21 there are $a' \in \bigcirc(\mathbb{IR})$ or $b' \in \bigcirc(\mathbb{IR})$ with $x = \epsilon' \overline{(a', 0, 0)}$ or $x = \epsilon' \overline{(0, b', 0)}$. This implies $\sigma = \epsilon' a' = \epsilon' |a'| \leq \epsilon' < \sigma$ or $\sigma = \epsilon' b' = \epsilon' |b'| \leq \epsilon' < \sigma$

and we get $\|x\| = \sigma$ and similarly $\|y\| = \sigma$. If $\alpha := (\frac{1}{2})^{\frac{1}{p}+1}$ and $\lambda \in [0, 1]$,

$(\lambda\alpha\sigma)^p + ((1-\lambda)\alpha\sigma)^p = \alpha^p\sigma^p(\lambda^p + (1-\lambda)^p) \leq 2(\frac{1}{2})^p\alpha^p\sigma^p \leq 2\alpha^p < 1$ holds, implying $\overline{(\alpha\sigma, 0, 0)} = \overline{(0, \alpha\sigma, 0)}$, thus $\alpha x = \alpha y$. $\beta \neq 0, \beta \in S$ is equivalent to $(\beta^{\frac{1}{p}}\sigma, 0, (1-\beta)^{\frac{1}{p}}\gamma) \sim (0, \beta^{\frac{1}{p}}\sigma, (1-\beta)^{\frac{1}{p}}\gamma)$. Define $f_\beta : [0, 1] \rightarrow \mathbb{R}$ by $f_\beta(\lambda) := \beta\sigma^p(\lambda^p + (1-\lambda)^p) + (1-\beta)\gamma^p$ ($\lambda \in [0, 1], \beta \in]0, 1]$). Because of 4.18, $\beta \in S$ is equivalent to $f_\beta(\lambda) < 1$ for $\lambda \in [0, 1]$. Hence $\beta \in S$ implies $\beta r^{1-p} + (1-\beta)\gamma^p = \beta\sigma^p(2(\frac{1}{2})^p) + (1-\beta)\gamma^p = f_\beta(\frac{1}{2}) < 1$. From $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = 2\sigma^{\frac{p}{1-p}} = r \geq 1$ and $\|z\| = \gamma < 1$ we conclude $\beta < \frac{1-\gamma^p}{r^{1-p}-\gamma^p}$ and by [4], 4.11(i) the assertion is proved. \square

4.22 raises the question if any intervall $[0, s[$, for arbitrary $0 < s < 1$, appears as a S for a suitable $D \in \mathbf{AC}_p$. \square

4.23 Definition: For $p < 1$, $\mathbb{K} := \mathbb{R}, \rho \in \mathbb{R}$, the relation $\hat{\wedge}$ on A (cf. 4.18) is defined by $(x_1, y_1, z_1) \hat{\wedge} (x_2, y_2, z_2)$ if and only if the following two conditions are fulfilled: (i) $z_1 = z_2$ and $y_1 + \rho x_1 = y_2 + \rho x_2$.

(ii) For all $\lambda \in [0, 1]$

$$|\lambda x_1 + (1-\lambda)x_2|^p + |\lambda y_1 + (1-\lambda)y_2|^p + |\lambda z_1 + (1-\lambda)z_2|^p \leq 1. \quad \square$$

Modifying the proof of 4.19, one easily gets the

4.24 Proposition: “ $\hat{\wedge}$ ” ($\rho \in \mathbb{R}$) is a congruence relation on A . \square

4.25 Lemma: If $p < 1$ then, for all $(x, y, 0) \in A$, there exist $a, b \in \bigcirc(\mathbb{R})$ with $ab = 0$ and $(x, y, 0) \hat{\wedge} (a, b, 0)$.

Proof: Write “ \sim ” := “ $\hat{\wedge}$ ”, consider the mappings $g, h : [0, 1] \rightarrow \mathbb{R}$ defined in 4.21, and replace $g(\lambda) < 1$ (resp. $h(\lambda) < 1$) in the proof of 4.21 by $g(\lambda) \leq 1$ (resp. $h(\lambda) \leq 1$) and the assertion follows. \square

Define $B_\rho := A/\hat{\wedge}$ ($\rho \in \mathbb{R}$), and let $\pi'_\rho : A \rightarrow B_\rho$ be the canonical projection. If no misunderstandings are possible one simply writes $\overline{(x, y, z)} := \pi'_\rho(\overline{(x, y, z)})$. \square

4.26 Theorem: For $p < 1$, $0 < \gamma < 1$, $1 \leq r \leq 2$, $\mathbb{K} := \mathbb{R}$ define $\sigma := (\frac{r}{2})^{\frac{1-p}{p}}$, $x := \sigma \bar{e}_1$, $y := \sigma \bar{e}_2$, $z := \gamma \bar{e}_3 \in B_1$. Then $\|x\| = \|y\| = \sigma$, $\|z\| = \gamma$, $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = r$ and for

$$S := \{\beta \in [0, 1] | \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z = \beta^{\frac{1}{p}}y + (1 - \beta)^{\frac{1}{p}}z\}$$

$$S = \left[0, \frac{1 - \|z\|^p}{(\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}})^{1-p} - \|z\|^p} \right] \text{ holds.}$$

Proof: Write “~”:= “ \triangleq ” for simplicity. The proof of the statements $\|x\| = \|y\| = \sigma$, $\|z\| = \gamma$ is similar to the corresponding proof in 4.22. Also $\alpha x = \alpha y$ for $\alpha := (\frac{1}{2})^{\frac{1}{p}+1}$, follows as in 4.22. For $\beta \in]0, 1]$, $\beta \in S$ is equivalent to $(\beta^{\frac{1}{p}}\sigma, 0, (1 - \beta)^{\frac{1}{p}}\gamma) \sim (0, \beta^{\frac{1}{p}}\sigma, (1 - \beta)^{\frac{1}{p}}\gamma)$. $f_\beta : [0, 1] \rightarrow \mathbb{R}$ is defined by $f_\beta(\lambda) := \beta\sigma^p(\lambda^p + (1 - \lambda)^p) + (1 - \beta)\gamma^p$ ($\lambda \in [0, 1]$, $\beta \in]0, 1]$).

Because of 4.23, $\beta \in S$ is equivalent to $f_\beta(\lambda) \leq 1$ for $\lambda \in [0, 1]$. Hence, in particular $\beta \in S$ implies $\beta r^{1-p} + (1 - \beta)\gamma^p = \beta\sigma^p(2(\frac{1}{2})^p) + (1 - \beta)\gamma^p = f_\beta(\frac{1}{2}) \leq 1$. From $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = 2\sigma^{\frac{p}{1-p}} = r \geq 1$ and $\|z\| = \gamma < 1$, $\beta \leq \frac{1 - \gamma^p}{r^{1-p} - \gamma^p}$ follows. $\beta_0 := \frac{1 - \gamma^p}{r^{1-p} - \gamma^p}$ yields $f_{\beta_0}(\lambda) = \beta_0\sigma^p(\lambda^p + (1 - \lambda)^p) + (1 - \beta_0)\gamma^p \leq 1$ for each $\lambda \in [0, 1]$, hence $\beta_0 \in S$. This proves the assertion ([4], 4.11(i)). \square

If $D \in \mathbf{TC}_p(\mathbf{AC}_p)$ and $y \in D$ one defines the mapping $\psi_y : D \rightarrow \{0, 1\}$ ($0 \neq 1$) by $\psi_y(x) := 1$ if there exist $z \in D$, $\beta \in]0, 1]$, $\gamma \in \mathbb{K}$, $|\gamma| = 1$, with $\gamma y = \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z$, and $\psi_y(x) := 0$, otherwise (cf. [3], p.3185). A relation “~” on D is defined by $a \sim b$ if and only if $a = b$ or $\psi_y(a) = \psi_y(b) = 0$.

4.27 Proposition: The relation “~” is a congruence relation on D .

Proof: Obviously, “~” is an equivalence relation. Let $a_i, b_i \in D$ with $a_i \sim b_i$ ($i \in \mathbb{N}$) and $\alpha \in \Omega_p(\Omega_{p,fin})$. If for all $i \in \text{supp } \alpha$ $a_i = b_i$, then $\sum_i \alpha_i a_i = \sum_i \alpha_i b_i$ holds. If there is an $i_0 \in \text{supp } \alpha$ with $a_{i_0} \neq b_{i_0}$ assume

$\psi_y(\sum_i \alpha_i a_i) = 1$. Then there exist $z \in D, \beta \in [0, 1], \gamma \in \mathbb{K}, |\gamma| = 1$ with

$$\begin{aligned} \gamma y &= \beta^{\frac{1}{p}} \sum_i \alpha_i a_i + (1 - \beta)^{\frac{1}{p}} z = (\beta^{\frac{1}{p}} |\alpha_{i_0}|) \left(\frac{\alpha_{i_0}}{|\alpha_{i_0}|} a_{i_0} \right) \\ &\quad + (1 - \beta |\alpha_{i_0}|^p)^{\frac{1}{p}} \left(\sum_{i \neq i_0} \frac{\beta^{\frac{1}{p}} \alpha_i}{(1 - \beta |\alpha_{i_0}|^p)^{\frac{1}{p}}} a_i + \frac{(1 - \beta)^{\frac{1}{p}}}{(1 - \beta |\alpha_{i_0}|^p)^{\frac{1}{p}}} z \right). \end{aligned}$$

(Here the bracket is to be read as *one* $\Omega_p(\Omega_{p,fin})$ -operation). This implies $\psi_y(\frac{\alpha_{i_0}}{|\alpha_{i_0}|} a_{i_0}) = 1$, thus $\psi_y(a_{i_0}) = 1$, contradicting $a_{i_0} \sim b_{i_0}, a_{i_0} \neq b_{i_0}$. Hence, necessarily $\psi_y(\sum_i \alpha_i a_i) = 0$ and, by symmetry, $\psi_y(\sum_i \alpha_i b_i) = 0$. This proves $\sum_i \alpha_i a_i \sim \sum_i \alpha_i b_i$ and “ \sim ” is a congruence relation. \square

4.28 Theorem: For $p < 1$ and $M \subset [0, 1[$ with $0 \in M$ there exist a p -totally convex space C and $x, y, z \in C$ with $\|z\| = \|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = 1$ (resp. $\|z\| = 1, \|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = 2$) such that for $S := \{\beta \in [0, 1] | \beta^{\frac{1}{p}} x + (1 - \beta)^{\frac{1}{p}} z = \beta^{\frac{1}{p}} y + (1 - \beta)^{\frac{1}{p}} z\}$ $S = M$ holds.

Proof: First let $M \neq [0, 1[$. Define $D := \widehat{\bigcap}_p \circ l_p(\{(0, 1), (1, 0)\})$ and put $x := (1, 0), y_\nu := (0, \nu), z := (0, 1) \in D$ with $\nu \in \{-1, 0\}$. For $\alpha \in M^c := [0, 1[\setminus M$ define $c_\alpha := \alpha^{\frac{1}{p}} x + (1 - \alpha)^{\frac{1}{p}} z$. The relation “ \sim ” on D is defined by $a \sim b$ if and only if $a = b$ or $\psi_{c_\alpha}(a) = \psi_{c_\alpha}(b) = 0$.

By 4.27, the relation $\sim := \bigcap_{\alpha \in M^c} \sim$ is a congruence relation on D . Denote D/\sim by C and by $[a]$ the set $\{b \in D | a \sim b\}$ ($a \in D$). Because of $M^c \subset]0, 1[, c_\alpha = \alpha^{\frac{1}{p}} x + (1 - \alpha)^{\frac{1}{p}} z, \psi_{c_\alpha}(x) = \psi_{c_\alpha}(z) = 1$ ($\alpha \in M^c$), $\|[x]\| = \|[z]\| = 1$ and $\|[x]\| = \|[z]\| = 1$ follows. From $-c_\alpha = \alpha^{\frac{1}{p}}(-1, 0) + (1 - \alpha)^{\frac{1}{p}}(0, -1)$ and $M^c \subset]0, 1[$ one gets $\psi_{c_\alpha}((0, -1)) = 1$ ($\alpha \in M^c$), hence $\|(0, -1)\| = 1$ and $\|[y_{-1}]\| = \|[0, -1]\| = 1$. This implies $\|[z]\| = \|[x]\|^{\frac{p}{1-p}} + \|[y_0]\|^{\frac{p}{1-p}} = 1$ and $\|[x]\|^{\frac{p}{1-p}} + \|[y_{-1}]\|^{\frac{p}{1-p}} = 2$. Put

$$S_\nu := \{\beta \in [0, 1] | \beta^{\frac{1}{p}} x + (1 - \beta)^{\frac{1}{p}} z \sim \beta^{\frac{1}{p}} y_\nu + (1 - \beta)^{\frac{1}{p}} z\}.$$

For all $\alpha \in M^c$ $\psi_{c_\alpha}(c_\alpha) = 1$ holds, thus $\alpha^{\frac{1}{p}} x + (1 - \alpha)^{\frac{1}{p}} z = c_\alpha \not\sim (0, 0)$. For every $\beta \in M \setminus \{0\}$ $|\beta^{\frac{1}{p}} \nu + (1 - \beta)^{\frac{1}{p}} 1| < 1$ holds, and $\|c_\alpha\| = 1$ implies

$(0,0) \sim (0, \beta^{\frac{1}{p}}\nu + (1-\beta)^{\frac{1}{p}}1) = \beta^{\frac{1}{p}}y_\nu + (1-\beta)^{\frac{1}{p}}z$ ([4], 4.5). This yields $M^c \cap S_\nu = \emptyset$. Now let $\beta \in M \setminus \{0\}$. Assume that there exists an $\alpha_0 \in M^c$ with $\psi_{c_{\alpha_0}}(\beta^{\frac{1}{p}}x + (1-\beta)^{\frac{1}{p}}z) = 1$. Then there are $(r,s) \in D, \gamma \in \mathbb{K}$ with $|\gamma| = 1$ and $\epsilon \in]0,1]$ with

$$\begin{aligned}\gamma c_{\alpha_0} &= \epsilon^{\frac{1}{p}}(\beta^{\frac{1}{p}}x + (1-\beta)^{\frac{1}{p}}z) + (1-\epsilon)^{\frac{1}{p}}(r,s) \\ &= ((\epsilon\beta)^{\frac{1}{p}} + (1-\epsilon)^{\frac{1}{p}}r, (\epsilon(1-\beta))^{\frac{1}{p}} + (1-\epsilon)^{\frac{1}{p}}s).\end{aligned}$$

$|\gamma| = 1, \|c_{\alpha_0}\| = 1$ implies $\|\gamma c_{\alpha_0}\| = 1$, hence $|r|^p + |s|^p = \|(r,s)\|^p = 1$. $\epsilon = 1$ leads to $\gamma c_{\alpha_0} = (\beta^{\frac{1}{p}}, (1-\beta)^{\frac{1}{p}})$, hence $\gamma \alpha_0^{\frac{1}{p}} = \beta^{\frac{1}{p}}$ and $\gamma = 1$, implying $\alpha_0 = \beta \in M^c \cap M = \emptyset$. Therefore $\epsilon < 1$ holds and one gets $1 = \|\gamma c_{\alpha_0}\| = |(\epsilon\beta)^{\frac{1}{p}} + (1-\epsilon)^{\frac{1}{p}}r|^p + |(\epsilon(1-\beta))^{\frac{1}{p}} + (1-\epsilon)^{\frac{1}{p}}s|^p \leq \epsilon\beta + (1-\epsilon)|r|^p + \epsilon(1-\beta) + (1-\epsilon)|s|^p = 1$, which implies $r = s = 0$, contradicting $|r|^p + |s|^p = 1$. Therefore, for all $\alpha \in M^c, \beta \in M \setminus \{0\}$ $\psi_{c_\alpha}(\beta^{\frac{1}{p}}x + (1-\beta)^{\frac{1}{p}}z) = 0$, consequently $\beta^{\frac{1}{p}}x + (1-\beta)^{\frac{1}{p}}z \sim (0,0) \sim \beta^{\frac{1}{p}}y_\nu + (1-\beta)^{\frac{1}{p}}z$ and $\beta \in S_\nu$. Obviously, $0 \in S_\nu$ holds, and $M \subset S_\nu$ is proved. The assumption $1 \in S_\nu$, i.e. $x \sim y_\nu$ implies the contradiction $(1,0) = x = y_\nu = (0,\nu)$ because of $\psi_{c_\alpha}(x) = 1$ ($\alpha \in M^c$). Hence $1 \notin S_\nu$ and $S_\nu = M$ ($\nu = -1, 0$) in case $M \neq [0,1[$ holds.

For $M = [0,1[$ define $C := L$ ([7], p. 985), L the t.c. Linton space, and put $z := 1, x := 1, y := 0 \in C$ (resp. $z := 1, x := 1, y := -1 \in C$), and the assertion is proved. \square

4.29 Theorem: Let $p < 1$ and $1 < r \leq 2$. Then there exist a real p-totally convex space $D, x, y \in D$ with $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = r$, $(\frac{1}{2})^{\frac{1}{p}}x = (\frac{1}{2})^{\frac{1}{p}}y$ and there are $z_n \in D$ with $\|z_n\| = 1$, such that for the sets $S_n := \{\beta \in [0,1] | \beta^{\frac{1}{p}}x + (1-\beta)^{\frac{1}{p}}z_n = \beta^{\frac{1}{p}}y + (1-\beta)^{\frac{1}{p}}z_n\}$ $S_n \neq \{0\}$, ($n \in \mathbb{N}$), and $\lim_{n \rightarrow \infty} \sup S_n = 0$ holds.

Proof: Put $\sigma := (\frac{r}{2})^{\frac{1-p}{p}}$ and $D := A_1 \in \mathbf{TC}_p$ (4.18, 4.19). Let $x := \overline{\sigma(1,0,0)}, y := \overline{\sigma(0,1,0)} \in A_1$. By 4.22, $\|x\| = \|y\| = \sigma$, which implies $\|x\|^{\frac{p}{1-p}} + \|y\|^{\frac{p}{1-p}} = r$. Let $z_i \in \bigcirc(\mathbb{R})$ ($i = 1, 2, 3$) with $0 < z_1, z_2 < 0$ and $|z_1|^p + |z_2|^p + |z_3|^p = 1$. Then we have $z := \overline{(z_1, z_2, z_3)} \in A_1$ and because

of $|z| = 1$ (4.18) $\|z\| = 1$ holds. Let “ \sim ” := “ γ ” (4.18) and

$$\begin{aligned} S &:= \{\beta \in [0, 1] | \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}z = \beta^{\frac{1}{p}}y + (1 - \beta)^{\frac{1}{p}}z\} \\ &= \{\beta \in [0, 1] | (\beta^{\frac{1}{p}}\sigma + (1 - \beta)^{\frac{1}{p}}z_1, (1 - \beta)^{\frac{1}{p}}z_2, (1 - \beta)^{\frac{1}{p}}z_3) \\ &\quad \sim ((1 - \beta)^{\frac{1}{p}}z_1, \beta^{\frac{1}{p}}\sigma + (1 - \beta)^{\frac{1}{p}}z_2, (1 - \beta)^{\frac{1}{p}}z_3)\}. \end{aligned}$$

For every $\beta \in]0, 1]$ $\beta \in S$ is equivalent with the following condition (*) because of $\beta^{\frac{1}{p}}\sigma \neq 0$: (*) For all $\lambda \in [0, 1]$

$$|(1 - \lambda)\beta^{\frac{1}{p}}\sigma + (1 - \beta)^{\frac{1}{p}}z_1|^p + |\lambda\beta^{\frac{1}{p}}\sigma + (1 - \beta)^{\frac{1}{p}}z_2|^p + (1 - \beta)|z_3|^p < 1.$$

First we assert $S \neq \{0\}$. For every $\lambda \in [0, 1]$ and $0 < \beta$, $\lambda\beta^{\frac{1}{p}}\sigma + (1 - \beta)^{\frac{1}{p}}z_2 < 0$ is equivalent with $\lambda < \lambda_0$, where $\lambda_0 := (\frac{1}{\beta} - 1)^{\frac{1}{p}}\frac{|z_2|}{\sigma}$. $\lambda_0 > 1$ holds if and only if $\beta < \frac{|z_2|^p}{|z_2|^p + \sigma^p}$. For a fixed β_0 with $0 < \beta_0 < \frac{|z_2|^p}{|z_2|^p + \sigma^p}$ a function $g : [0, 1] \rightarrow \mathbb{R}$ is defined by $g(t) := \left((1 - t)\beta_0^{\frac{1}{p}}\sigma + (1 - \beta_0)^{\frac{1}{p}}z_1 \right)^{\frac{1}{p}} + \left(-\left(t\beta_0^{\frac{1}{p}}\sigma + (1 - \beta_0)^{\frac{1}{p}}z_2 \right) \right)^p + (1 - \beta_0)|z_3|^p$ ($t \in [0, 1]$). g is differentiable in $]0, 1[$ with

$$\begin{aligned} g'(t) &= -p\beta_0^{\frac{1}{p}}\sigma \left(\left((1 - t)\beta_0^{\frac{1}{p}}\sigma + (1 - \beta_0)^{\frac{1}{p}}z_1 \right)^{p-1} \right. \\ &\quad \left. + \left(-\left(t\beta_0^{\frac{1}{p}}\sigma + (1 - \beta_0)^{\frac{1}{p}}z_2 \right) \right)^{p-1} \right). \end{aligned}$$

Thus $g'(t) < 0$ for all $t \in]0, 1[$ and $g(0) = \left(\beta_0^{\frac{1}{p}}\sigma + (1 - \beta_0)^{\frac{1}{p}}z_1 \right)^p + (1 - \beta_0)(|z_2|^p + |z_3|^p) < \beta_0\sigma^p + (1 - \beta_0)z_1^p + (1 - \beta_0)(|z_2|^p + |z_3|^p) = \beta_0\sigma^p + (1 - \beta_0) \leq 1$ holds implying $0 \leq g(t) < 1$ for all $t \in [0, 1]$. Thus β_0 fulfills (*) and $S \neq \{0\}$ follows. $\left(\frac{\gamma}{2}\right)^{\frac{1}{p}}x = \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\gamma^{\frac{1}{p}}x + (1 - \gamma)^{\frac{1}{p}}z\right) - \left(\frac{1}{2}\right)^{\frac{1}{p}}\left((1 - \gamma)^{\frac{1}{p}}z\right) = \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\gamma^{\frac{1}{p}}y + (1 - \gamma)^{\frac{1}{p}}z\right) - \left(\frac{1}{2}\right)^{\frac{1}{p}}\left((1 - \gamma)^{\frac{1}{p}}z\right) = \left(\frac{\gamma}{2}\right)^{\frac{1}{p}}y$ ($\gamma \in S \setminus \{0\}$) yields $\left(\frac{1}{2}\right)^{\frac{1}{p}}x = \left(\frac{1}{2}\right)^{\frac{1}{p}}y$ because of [4], 4.14. Now, for $\beta \in]0, 1[$ put $t' := \frac{1}{2} + \left(\frac{1}{\beta} - 1\right)^{\frac{1}{p}}\frac{z_1}{\sigma}$ and $s(z_1) := \frac{2^p z_1^p}{2^p z_1^p + \sigma^p}$. Obviously, $0 < t'$ holds and $t' < 1$ is equivalent with $s(z_1) < \beta$. Put $z_2 := -z_1 < 0$. If $\beta \in S$ with $s(z_1) < \beta$, then

$0 < t' < 1$ follows and $(*)$ is fulfilled for $\lambda = t'$. $1 = \sum_{i=1}^3 |z_i|^p$ and $0 < z_1$ imply $|z_3| < 1 < r$, hence $0 < r^{1-p} - |z_3|^p$ holds. Application of $(*)$ to $\lambda = t'$ leads to $\beta < s(z_3) := \frac{1-|z_3|^p}{r^{1-p}-|z_3|^p}$ by an elementary computation. Consequently, $\beta \in S$ implies $\beta \leq \max\{s(z_1), s(z_3)\}$.

Now, for every $n \in \mathbb{N}$, we choose an element $z_n := \overline{(z_{1,n}, z_{2,n}, z_{3,n})} \in D$ with $0 < z_{1,n} < 1$, $0 \leq z_{3,n} < 1$, $z_{2,n} := -z_{1,n}$, $\sum_{i=1}^3 |z_{i,n}|^p = 1$; furthermore, the sequences $(z_{1,n})_{n \in \mathbb{N}}$ and $(z_{3,n})_{n \in \mathbb{N}}$ are supposed to be convergent with $\lim_{n \rightarrow \infty} z_{1,n} = 0$ and $\lim_{n \rightarrow \infty} z_{3,n} = 1$. For the sets $S_n := \{\beta \in [0, 1] \mid \beta^{\frac{1}{p}}x + (1-\beta)^{\frac{1}{p}}z_n = \beta^{\frac{1}{p}}y + (1-\beta)^{\frac{1}{p}}z_n\}$ we have $S_n \neq \{0\}$ ($n \in \mathbb{N}$) and $(\frac{1}{2})^{\frac{1}{p}}x = (\frac{1}{2})^{\frac{1}{p}}y$ holds. Because of $\lim_{n \rightarrow \infty} z_{1,n} = 0$ and $\lim_{n \rightarrow \infty} z_{3,n} = 1$, $\lim_{n \rightarrow \infty} s(z_{1,n}) = 0$ and $\lim_{n \rightarrow \infty} s(z_{3,n}) = 0$ follows. This leads to $\lim_{n \rightarrow \infty} \max\{s(z_{1,n}), s(z_{3,n})\} = 0$. As seen above, $\beta \in S_n$ implies $\beta \leq \max\{s(z_{1,n}), s(z_{3,n})\}$ ($n \in \mathbb{N}$) hence we get $\lim_{n \rightarrow \infty} \sup S_n = 0$. \square

Let $(D_i, \|\cdot\|_i)_{i \in I}$ be a family of p -absolutely convex subspaces of $D \in \mathbf{AC}_p$ and put $C := \bigcap_{i \in I} D_i$. In case $p = 1$ $\|x\|_C = \sup\{\|x\|_i \mid i \in I\}$ holds for all $x \in C$ ([7], 6.6). For $p < 1$ one has the following

4.30 Proposition: For $p < 1$ $\sup\{\|x\|_i \mid i \in I\} \leq \|x\|_C$ ($x \in C$) holds, but in general (for $\mathbb{K} := \mathbb{R}$) $\sup\{\|x\|_i \mid i \in I\} \neq \|x\|_C$ for some $x \in C$.

Proof: Let $in_i : C \hookrightarrow D_i$ be the inclusion of C in D_i ($i \in I$). For every $x \in C$ $\|x\|_i = \|in_i(x)\|_i \leq \|x\|_C$ ($i \in I$) holds ([4], 4.3(ii)), thus $\sup\{\|x\|_i \mid i \in I\} \leq \|x\|_C$. Put $D := A_1 \in \mathbf{TC}_p$ (4.18, 4.19). For all $x \in \overset{\circ}{\bigcup}(\mathbb{R})$, $(x, 0, 0) \sim (0, x, 0)$ is equivalent by 4.18 with $|x|^p(\lambda^p + (1-\lambda)^p) < 1$ for all $\lambda \in [0, 1]$. The mapping $g : [0, 1] \rightarrow \mathbb{R}$, defined by $g(t) := t^p + (1-t)^p$ ($t \in [0, 1]$) has an absolute maximum in $t = \frac{1}{2}$ with $g(\frac{1}{2}) = (\frac{1}{2})^{p-1}$. Hence $(x, 0, 0) \sim (0, x, 0)$ is equivalent with $|x| < \omega_p$ for all $x \in \overset{\circ}{\bigcup}(\mathbb{R})$.

$C_1 := \overline{\overset{\circ}{\bigcup}_p(\mathbb{R})(0, 1, 0)}$ and $C_2 := \overline{\overset{\circ}{\bigcup}_p(\mathbb{R})(1, 0, 0)}$ are p -totally convex subspaces of D , and $z := (0, (\frac{1}{2})^{\frac{1}{p}}, 0) = ((\frac{1}{2})^{\frac{1}{p}}, 0, 0) \in C_1 \cap C_2$ holds. By the above considerations we get $C_1 \cap C_2 = \overset{\circ}{\bigcup}(\mathbb{R})\overline{(0, \omega_p, 0)}$. For every $\gamma \in \overset{\circ}{\bigcup}(\mathbb{R})$ with $z = \gamma\overline{(0, \omega_p, 0)}$ 4.18 yields $\frac{1}{2}\omega_p = (\frac{1}{2})^{\frac{1}{p}} = \gamma\omega_p$, hence $\gamma = \frac{1}{2}$. Now, a simple

computation shows $\|z\|_{C_1 \cap C_2} = \frac{1}{2} > \sup \{\|z\|_{C_i} | i = 1, 2\} = (\frac{1}{2})^{\frac{1}{p}}$. \square

4.31 Definition: Let D be a p-absolutely convex space and $x, y \in D$.

- (i) The sets $]x, y[:= \{\beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}y | 0 < \beta < 1\}$, $[x, y[:= \{\beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}y | 0 < \beta \leq 1\}$, $]x, y] := \{\beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}y | 0 \leq \beta < 1\}$, $[x, y] := \{\beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}y | 0 \leq \beta \leq 1\}$ are the so-called open (right open, left open, closed) segment between x and y .
- (ii) $\partial D := \{z \in D | \|z\| = 1\}$ is the boundary of D (cf. [9], p.1473).
- (iii) $\overset{\circ}{D} := D \setminus \partial D$ is the interior of D ([8], 10.1). \square

4.32 Proposition: Let D be a p-absolutely convex space and $x, y \in D$. Then the following statements hold:

- (i) For $p = 1$ $]x, y[\subset \overset{\circ}{D}$ or $]x, y[\subset \partial D$.
- (ii) For $p < 1$, $x_0, y_0 \in [x, y[$ or $x_0, y_0 \in]x, y]$ implies $]x_0, y_0[\in \overset{\circ}{D}$.

Proof: (i) Let $a \in]x, y[$ with $\|a\| < 1$. Then there exists an α with $0 < \alpha < 1$ and $a = \alpha x + (1 - \alpha)y$. For every $b \in]x, y[$ there exists a β with $0 < \beta < 1$ and $b = \beta x + (1 - \beta)y$. In case $\beta \geq \alpha$ for $\gamma := \frac{\beta - \alpha}{1 - \alpha} \in [0, 1[$ one gets $\gamma x + (1 - \gamma)a = \frac{\beta - \alpha}{1 - \alpha}x + \frac{1 - \beta}{1 - \alpha}(\alpha x + (1 - \alpha)y) = \beta x + (1 - \beta)y = b$ and $\|b\| \leq \gamma\|x\| + (1 - \gamma)\|a\| < 1$ holds by [4], 4.5. In case $\beta < \alpha$ put $\gamma := \frac{\alpha - \beta}{\alpha} \in]0, 1[$. Then $\gamma y + (1 - \gamma)a = \frac{\alpha - \beta}{\alpha}y + \frac{\beta}{\alpha}(\alpha x + (1 - \alpha)y) = \beta x + (1 - \beta)y = b$ follows, and, by [4], 4.5, $\|b\| < 1$. Thus $]x, y[\subset \overset{\circ}{D}$ or $]x, y[\subset \partial D$.

(ii) Because of the assumption there exist $\beta, \gamma \in [0, 1]$ with $(\beta, \gamma) \neq (1, 0)$, $(0, 1)$ and $x_0 = \beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}y$, $y_0 = \gamma^{\frac{1}{p}}x + (1 - \gamma)^{\frac{1}{p}}y$. Let $a \in]x_0, y_0[$ thus $a = \alpha^{\frac{1}{p}}x_0 + (1 - \alpha)^{\frac{1}{p}}y_0$ for some $\alpha \in]0, 1[$. From [4], 4.5 one gets

$$\begin{aligned}\|a\|^p &= \|\alpha^{\frac{1}{p}}(\beta^{\frac{1}{p}}x + (1 - \beta)^{\frac{1}{p}}y) + (1 - \alpha)^{\frac{1}{p}}(\gamma^{\frac{1}{p}}x + (1 - \gamma)^{\frac{1}{p}}y)\|^p \\ &= \|((\alpha\beta)^{\frac{1}{p}} + ((1 - \alpha)\gamma)^{\frac{1}{p}})x + ((\alpha(1 - \beta))^{\frac{1}{p}} + ((1 - \alpha)(1 - \gamma))^{\frac{1}{p}})y\|^p \\ &\leq ((\alpha\beta)^{\frac{1}{p}} + ((1 - \alpha)\gamma)^{\frac{1}{p}})^p + ((\alpha(1 - \beta))^{\frac{1}{p}} + ((1 - \alpha)(1 - \gamma))^{\frac{1}{p}})^p.\end{aligned}$$

Since in all cases $(0 < \alpha\beta$ and $0 < (1 - \alpha)\gamma)$ or $(0 < \alpha(1 - \beta)$ and $0 < (1 - \alpha)(1 - \gamma))$ holds, $\|a\|^p < \alpha\beta + (1 - \alpha)\gamma + \alpha(1 - \beta) + (1 - \alpha)(1 - \gamma) = 1$

follows i.e. $a \in \overset{\circ}{D}$. This implies $]x_0, y_0[\subset \overset{\circ}{D}$. \square

Certain subsets of the boundary of an absolutely convex space and the interior of such spaces are convex spaces ([10], 1.8, [9], §2). In case $p < 1$, $D \in \mathbf{AC}_p$, for every $y \in \partial D$, $(\frac{1}{2})^{\frac{1}{p}}y + (\frac{1}{2})^{\frac{1}{p}}y = \omega_p y \notin \partial D$ holds, so that the boundary of D is not accessible by the methods of [9], §2. Furthermore, 4.32(ii), applied to the p -absolutely convex space $D := \Omega_{p,fin}$, $p < 1$, shows (with $\Omega'_{p,fin} := \{\alpha = (\alpha_i)_{i \in \mathbb{N}} \in [0,1]^{\mathbb{N}} \mid |\text{supp } \alpha| < \infty \text{ and } \sum_i \alpha_i^p = 1\}$) that there do not exist “ p -convex spaces” in the sense of [4], 2.4. \square

4.33 Lemma: For $D \in \mathbf{AC}_p$ let “ \sim ” be a congruence relation on D . Then, for all $a, b \in D$, and all $\alpha \in \overset{\circ}{\bigcirc}(\mathbb{K}) \setminus \{0\}$, $\alpha a - \alpha b \sim 0$ implies $\beta a \sim \beta b$ for all $\beta \in \overset{\circ}{\bigcirc}(\mathbb{K})$.

Proof: From $((\frac{1}{4})^{\frac{1}{p}}\alpha)b = (\frac{1}{4})^{\frac{1}{p}}0 + (\frac{1}{4})^{\frac{1}{p}}(\alpha b) \sim (\frac{1}{4})^{\frac{1}{p}}(\alpha a - \alpha b) + (\frac{1}{4})^{\frac{1}{p}}(\alpha b) = ((\frac{1}{4})^{\frac{1}{p}}\alpha)a$ and $\alpha \neq 0$ the assertion follows by [4], 4.14. \square

If no misunderstandings are possible, we will simply write $\overset{\circ}{\bigcirc}(\mathbb{K})$ instead of $\overset{\circ}{\bigcirc}_p(\mathbb{K})$, resp. $\overset{\circ}{\bigcirc}_{fin}(\mathbb{K})$ instead of $\overset{\circ}{\bigcirc}_{p,fin}(\mathbb{K})$ in the following. \square

4.34 Theorem (cf. [7], 4.4): The p -totally (p -absolutely) convex space $\overset{\circ}{\bigcirc}(\mathbb{K})$ ($\overset{\circ}{\bigcirc}_{fin}(\mathbb{K})$) permits for $p = 1$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, precisely three, for $p < 1$, $\mathbb{K} = \mathbb{R}$, precisely four, and for $p < 1$, $\mathbb{K} = \mathbb{C}$, infinitely many congruence relations.

Proof: First let $p \leq 1$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and “ \sim ” be a congruence relation on $\overset{\circ}{\bigcirc}(\mathbb{K}) \in \mathbf{TC}_p$. For $a, b \in \overset{\circ}{\bigcirc}(\mathbb{K})$ with $a \sim b$ and $a \neq b$, $(\frac{1}{2})^{\frac{1}{p}}a - (\frac{1}{2})^{\frac{1}{p}}b \in \overset{\circ}{\bigcirc}(\mathbb{K}) \setminus \{0\}$ follows and $((\frac{1}{2})^{\frac{1}{p}}a - (\frac{1}{2})^{\frac{1}{p}}b)1 = (\frac{1}{2})^{\frac{1}{p}}a - (\frac{1}{2})^{\frac{1}{p}}b \sim (\frac{1}{2})^{\frac{1}{p}}a - (\frac{1}{2})^{\frac{1}{p}}a = 0$ implies $S \neq \{0\}$ for the set $S := \{\lambda \in \overset{\circ}{\bigcirc}(\mathbb{K}) \mid \lambda 1 \sim \lambda 0\}$. By [4], 4.17(ii), applied to $\overset{\circ}{\bigcirc}(\mathbb{K})/\sim$, we have $S = \overset{\circ}{\bigcirc}(\mathbb{K})$ or $S = \overset{\circ}{\bigcirc}(\mathbb{K})$. $S = \overset{\circ}{\bigcirc}(\mathbb{K})$ means $x \sim y$ for all $x, y \in \overset{\circ}{\bigcirc}(\mathbb{K})$. If $S = \overset{\circ}{\bigcirc}(\mathbb{K})$, for every $x \in \overset{\circ}{\bigcirc}(\mathbb{K})$ with $|x| = 1$, obviously $x \not\sim 0$ holds. Hence, for $p = 1$ $a \sim b$ with $|a| = |b| = 1$ leads to $1 = \frac{1}{b}b \sim \frac{1}{b}a = \frac{a}{b}$ and $1 = \frac{b}{2a}\frac{a}{b} + \frac{1}{2}1 \sim \frac{b}{2a}1 + \frac{1}{2}1 = \frac{1}{2}(\frac{b}{a} + 1)$. This implies

$a = b$, since $\frac{1}{2}(\frac{b}{a} + 1) \in \overset{\circ}{\bigcirc}(\mathbb{K})$ otherwise. Thus the equivalence classes in case $S = \overset{\circ}{\bigcirc}(\mathbb{K})$ for $p = 1$ are precisely $\overset{\circ}{\bigcirc}(\mathbb{K})$ and $\{x\}$ ($x \in \widehat{\bigcirc}(\mathbb{K})$ with $|x| = 1$).

Now, for every $p \leq 1$ the relation “ \approx ” on $\widehat{\bigcirc}(\mathbb{K})$, defined by $x \approx y$ if and only if $x = y$ or $x, y \in \overset{\circ}{\bigcirc}(\mathbb{K})$ obviously is an equivalence relation on $\widehat{\bigcirc}(\mathbb{K})$. Let $x_i, y_i \in \widehat{\bigcirc}(\mathbb{K})$ with $x_i \approx y_i$ ($i \in \mathbb{N}$), $\alpha \in \Omega_p$. In case $x_i = y_i$ for all $i \in \text{supp } \alpha$, we get $\sum_i \alpha_i x_i = \sum_i \alpha_i y_i$. So, let $i_0 \in \text{supp } \alpha$ with $x_{i_0} \neq y_{i_0}$. Because of $\|x_{i_0}\|, \|y_{i_0}\| < 1$, $\|\sum_i \alpha_i x_i\| \leq (\sum_i |\alpha_i|^p \|x_i\|^p)^{\frac{1}{p}} < 1$ follows ([4], 4.5) and, by symmetry, $\|\sum_i \alpha_i y_i\| < 1$, thus $\sum_i \alpha_i x_i \approx \sum_i \alpha_i y_i$, and “ \approx ” is a congruence relation on $\widehat{\bigcirc}(\mathbb{K})$. This means that there are precisely three congruence relations on $\widehat{\bigcirc}(\mathbb{K})$ for $p \leq 1$ and at last three for $p < 1$.

Now let $p < 1$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and let “ \sim ” be an *equivalence* relation on $\widehat{\bigcirc}(\mathbb{K})$ with $x \sim y$ for all $x, y \in \overset{\circ}{\bigcirc}(\mathbb{K})$, and the following property: For all $a, b \in \widehat{\bigcirc}(\mathbb{K})$ and all $\alpha \in \bigcirc(\mathbb{K})$ with $|\alpha| = 1$, $a \sim b$ implies $\alpha a \sim \alpha b$. We show that “ \sim ” is a congruence relation on $\widehat{\bigcirc}(\mathbb{K})$. For this let $a_i, b_i \in \widehat{\bigcirc}(\mathbb{K})$ with $a_i \sim b_i$ ($i \in \mathbb{N}$) and $\alpha \in \Omega_p$. If there is an $i_0 \in \text{supp } \alpha$ with $0 < |\alpha_{i_0}| < 1$, $|\sum_i \alpha_i a_i| \leq \sum_i |\alpha_i| |a_i| \leq \sum_i |\alpha_i| < \sum_i |\alpha_i|^p \leq 1$ follows and by symmetry $|\sum_i \alpha_i b_i| < 1$, hence $\sum_i \alpha_i a_i \sim \sum_i \alpha_i b_i$. This is also trivially fulfilled for $\alpha = 0$, and in case $|\alpha_{i_0}| = 1$ for some $i_0 \in \text{supp } \alpha$ by assumption $\alpha_{i_0} a_{i_0} \sim \alpha_{i_0} b_{i_0}$, hence $\sum_i \alpha_i a_i = \alpha_{i_0} a_{i_0} \sim \alpha_{i_0} b_{i_0} = \sum_i \alpha_i b_i$. Consequently, “ \sim ” is a congruence relation on $\widehat{\bigcirc}(\mathbb{K})$.

In case $p < 1$, $\mathbb{K} = \mathbb{R}$, the relation “ \sim ” on $\widehat{\bigcirc}(\mathbb{R})$ defined by $a \sim b$ if and only if $a, b \in \overset{\circ}{\bigcirc}(\mathbb{R})$ or $|a| = |b| = 1$ ($a, b \in \widehat{\bigcirc}(\mathbb{R})$) is, as seen above, the last possibility for a congruence relation on $\widehat{\bigcirc}(\mathbb{R})$. Obviously, “ \sim ” is an equivalence relation on $\widehat{\bigcirc}(\mathbb{R})$ and $a \sim b$ implies $\alpha a \sim \alpha b$ for all $a, b \in \widehat{\bigcirc}(\mathbb{R})$, $\alpha \in \bigcirc(\mathbb{R})$ with $|\alpha| = 1$. By the above result, “ \sim ” is a congruence relation on $\widehat{\bigcirc}(\mathbb{R})$, hence there are precisely four congruence relations on $\widehat{\bigcirc}(\mathbb{R})$, $p < 1$.

Obviously, the relation “ \sim ” on $\widehat{\bigcirc}(\mathbb{C})$, defined by $a \sim b$ if and only if $a, b \in \overset{\circ}{\bigcirc}(\mathbb{C})$ or $|a| = |b| = 1$ ($a, b \in \widehat{\bigcirc}(\mathbb{C})$) is for $p < 1$ a congruence relation on $\widehat{\bigcirc}(\mathbb{C})$. Finally, we have to show that $\widehat{\bigcirc}(\mathbb{C})$, for $p < 1$, permits infinitely

many congruence relations. Take any $m \in \mathbb{N}$. A relation “ \sim ” on $\widehat{\bigcirc}(\mathbb{C})$ is defined by $a \sim b$ if and only if there exists a $r \in \mathbb{N}_0$ with $a = \zeta_m^r b$ ($a, b \in \widehat{\bigcirc}(\mathbb{C})$). Obviously, “ \sim ” is an equivalence relation on $\widehat{\bigcirc}(\mathbb{C})$. Let $\alpha \in \bigcirc(\mathbb{C})$ with $|\alpha| = 1$, and $a, b \in \widehat{\bigcirc}(\mathbb{C})$ with $a \sim b$. Then there is a $r \in \mathbb{N}_0$ with $a = \zeta_m^r b$. This implies $\alpha a = \zeta_m^r (\alpha b)$, hence $\alpha a \sim \alpha b$. Hence, “ \sim ” is a congruence relation on $\widehat{\bigcirc}(\mathbb{C})$. For all $m, n \in \mathbb{N}$ with $m < n$ there does not exist a $s \in \mathbb{N}_0$ with $\zeta_n = \zeta_m^s$. Thus we get infinitely many congruence relations on $\widehat{\bigcirc}(\mathbb{C})$ for $p < 1$. The proof for $\widehat{\bigcirc}_{fin}(\mathbb{K}) \in \mathbf{AC}_p$ is similar. \square

4.35 Corollary: If one defines, for $p \leq 1$ and $x_i \in \{0, 1\}$ ($0 \neq 1$), $i \in \mathbb{N}$, $\sum_i \alpha_i x_i := 1$, if there is an $i_0 \in \mathbb{N}$ with $|\alpha_{i_0}| = 1$ and $x_{i_0} = 1$, and $\sum_i \alpha_i x_i := 0$ otherwise, then $\{0, 1\}$ becomes a p -totally convex space. \square

4.36 Definition (cf. [7], p.985): Let $p \leq 1$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The relation “ \sim ” on $\widehat{\bigcirc}(\mathbb{K})$ defined by $x \sim y$ if and only if $x, y \in \overset{\circ}{\bigcirc}(\mathbb{K})$ or $x = y$ ($x, y \in \widehat{\bigcirc}(\mathbb{K})$) is a congruence relation on $\widehat{\bigcirc}(\mathbb{K})$ (cf. 4.34). The quotient $L_{\mathbb{K}} := \widehat{\bigcirc}(\mathbb{K})/\sim$ is the so-called Linton space with $|L_{\mathbb{R}}| = 3$ and $|L_{\mathbb{C}}| = \infty$. \square

4.37 Corollary (cf. [4], 1.4, 1.6): The functors $\bigcirc_p : \mathbf{Ban}_p \rightarrow \mathbf{Set}$ and $\bigcirc_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{Set}$ are not monadic. \square

4.38 Corollary: Let $D \neq \{0\}$ be an absolutely convex space. Then, for any $x \in D \setminus \{0\}$, and $\alpha, \beta \in \mathbb{K}$ with $|\alpha| = |\beta| = 1$, $\alpha x = \beta x$ implies $\alpha = \beta$.

Proof: Obviously, the mapping $f : \widehat{\bigcirc}(\mathbb{K}) \rightarrow D$, defined by $f(\lambda) := \lambda x$ ($\lambda \in \widehat{\bigcirc}(\mathbb{K})$) is an \mathbf{AC} -morphism. Denote by “ \sim ” the congruence relation on $\widehat{\bigcirc}(\mathbb{K})$ induced by f . Let $\alpha, \beta \in \bigcirc(\mathbb{K})$ with $|\alpha| = |\beta| = 1$ and $\alpha x = \beta x$. Then $\alpha \sim \beta$ and, since $x \neq 0$, $1 \not\sim 0$ holds. $(\frac{\alpha}{2} - \frac{\beta}{2})x = \frac{1}{2}(\alpha x) - \frac{1}{2}(\beta x) = 0$ implies $S \neq \{0\}$ for the set $S := \{\gamma \in \bigcirc(\mathbb{K}) \mid \gamma x = 0\}$, if $\alpha \neq \beta$. Because of $1 \not\sim 0$ we then get $S = \overset{\circ}{\bigcirc}(\mathbb{K})$ ([4], 4.17(ii)). Now, 4.34 yields $\alpha = \beta$. \square

4.38 does not hold for $p < 1$, since in $\{0, 1\}$ (4.35) we have $(-1) \cdot 1 = 1 \cdot 1$. \square

4.39 Corollary ([7], 4.7): Let $D \neq \{0\}$ be a complex absolutely convex space. Then $\text{card}(D) \geq \text{card}(\mathbb{C})$.

Proof: Let $x \in D \setminus \{0\}$. The mapping $f : \widehat{\bigcirc}(\mathbb{C}) \rightarrow D$ defined by $f(\lambda) := \lambda x$ is injective on $\partial \widehat{\bigcirc}(\mathbb{C})$ (4.38). This implies $\text{card}(D) \geq \text{card}(\mathbb{C})$. \square

4.40 Corollary ([7], p.986): Let D be a real absolutely convex space with $\text{card}(D) < \infty$. Then $\text{card}(D)$ is odd.

Proof: Let $x \in D \setminus \{0\}$. By 4.38, $x \neq -x$ holds, and we are finished. \square

4.35 shows that 4.40 does not hold for $p < 1$. \square

§5 The Left Adjoint of $\widehat{\bigcirc}_p : \text{Ban}_p \rightarrow \text{TC}_p$

It is known (cf. e.g. [6], 3.7) that $\widehat{\bigcirc}_p$ has a left adjoint S_p . We will now construct S_p explicitly. For this let D be a p -totally convex space. On the set $\mathbb{K} \times D$ one defines an equivalence relation “ \sim ” by putting $(\lambda_0, d_0) \sim (\lambda_1, d_1)$ if and only if there is a $\lambda \geq \max\{|\lambda_0|, |\lambda_1|\}$ with $\frac{\lambda_0}{\lambda} d_0 = \frac{\lambda_1}{\lambda} d_1$. The equivalence class of (λ, x) is denoted by $\overline{(\lambda, x)}$, and the set of equivalence classes of $\mathbb{K} \times D$ by $S_p(D)$. There is a canonical mapping $\sigma'_D : D \rightarrow S_p(D)$, defined by $\sigma'_D(x) := \overline{(1, x)} (x \in D)$. Next one introduces scalar-multiplication and addition on $S_p(D)$ by $\mu \overline{(\lambda, x)} := \overline{(\mu \lambda, x)} ((\lambda, x) \in \mathbb{K} \times D, \mu \in \mathbb{K})$ resp. $\overline{(\lambda_0, d_0)} + \overline{(\lambda_1, d_1)} := \overline{(\lambda, \frac{\lambda_0}{\lambda} d_0 + \frac{\lambda_1}{\lambda} d_1)}$ where $\lambda > (|\lambda_0|^p + |\lambda_1|^p)^{\frac{1}{p}}$. Obviously, the scalar-multiplication and, by a simple computation, the addition on $S_p(D)$ is well-defined and $S_p(D)$ is a \mathbb{K} -vector space. Furthermore, for $(\lambda, d) \in S_p(D)$ one puts $\|\overline{(\lambda, d)}\| := \inf\{|\mu| \mid \mu \in \mathbb{K} \text{ and } \overline{(\lambda, d)} = \mu \overline{(1, y)} \text{ for some } y \in D\}$. Then one has the following

5.1 Proposition: For $D \in \text{TC}_p$, $S_p(D)$ is a p -normed \mathbb{K} -vector space.

Proof: $\|\overline{(\lambda, d)}\| \geq 0$ is trivial. For $(\lambda, d) \in \mathbb{K} \times D$ with $\|\overline{(\lambda, d)}\| = 0$, for $\mu := (\frac{1}{2})^{\frac{1}{p}}$ there is a $y \in D$ with $\overline{(\lambda, d)} = \overline{(\mu^2, y)} = \overline{(\mu, \mu y)}$, hence $\|\overline{(\mu, \mu y)}\| = 0$

Let $\varepsilon \in]0, \omega_p[$. Then there exists a $z \in D$ with $\overline{(\mu, \mu y)} = \overline{(\varepsilon\mu, z)}$. Consequently, there is a $\sigma > \mu$ with $\frac{\mu}{\sigma}(\mu y) = \frac{\varepsilon\mu}{\sigma}z = \frac{\varepsilon}{\sigma}(\varepsilon z)$, implying $\mu y = \varepsilon z$ because of $\|\mu y\| < \omega_p$ and $\|\varepsilon z\| < \omega_p$ ([4], 4.14). This leads to $\|\mu y\| = 0$ and, by [4], 4.15, to $\mu y = 0$, hence $\overline{(\lambda, d)} = 0$. Obviously, $\|-\|$ is homogenous. The proof of the p-triangle inequality is a simple computation. \square

5.2 Lemma: The following statements hold for every $D \in \mathbf{TC}_p$.

- (i) $\|\sigma'_D(x)\| \leq \|x\| (x \in D)$.
- (ii) For $\alpha \in \Omega_{p,fin}$, $x_i \in D (i \in \mathbb{N})$ $\sigma'_D(\sum_i \alpha_i x_i) = \sum_i \alpha_i \sigma'_D(x_i)$ holds.
- (iii) For $\alpha \in \Omega_p$, $(x_i)_{i \in \mathbb{N}} \in D^{\mathbb{N}}$ and any $n \in \mathbb{N}$ one has $\sigma'_D(\sum_i \alpha_i x_i) = \sum_{i=1}^n \alpha_i \sigma'_D(x_i) + \sigma'_D(\sum_{i=n+1}^{\infty} \alpha_i x_i)$.

Proof: (i) $x = \lambda x_0$ implies $\sigma'_D(x) = \lambda \sigma'_D(x_0)$, hence $\|\sigma'_D(x)\| \leq \|x\|$.

(ii) This follows from the definition of the addition and the scalar-multiplication on $S_p(D)$.

(iii) Put $x := \sum_{i=1}^n \alpha_i x_i$ and $y := \sum_{i=n+1}^{\infty} \alpha_i x_i$. Then one gets by (ii)

$$\begin{aligned} \sigma'_D(\sum_i \alpha_i x_i) &= 2\sigma'_D\left(\frac{1}{2}x + \frac{1}{2}y\right) = \sigma'_D\left(\sum_{i=1}^n \alpha_i x_i\right) + \sigma'_D\left(\sum_{i=n+1}^{\infty} \alpha_i x_i\right) \\ &= \sum_{i=1}^n \alpha_i \sigma'_D(x_i) + \sigma'_D\left(\sum_{i=n+1}^{\infty} \alpha_i x_i\right). \end{aligned} \quad \square$$

5.3 Lemma: For $D \in \mathbf{TC}_p$ $\overset{\circ}{\bigcirc}_p(S_p(D)) \subset \sigma'_D(D) \subset \bigcirc_p(S_p(D))$ holds.

Proof: The second inclusion follows from 5.2(i). For $\overline{(\lambda, d)} \in \overset{\circ}{\bigcirc}_p(S_p(D))$ there are $\mu \in \overset{\circ}{\bigcirc}(\mathbb{K})$, $y \in D$ with $\overline{(\lambda, d)} = \overline{(\mu, y)} = \sigma'_D(\mu y) \in \sigma'_D(D)$. \square

5.4 Proposition: For any $D \in \mathbf{TC}_p$, $S_p(D)$ is a p-Banach space.

Proof: By 5.1, $S_p(D)$ is a p-normed \mathbb{K} -vector space. Let $z_i \in S_p(D) (i \in \mathbb{N})$ with $\sum_i \|z_i\|^p < \infty$. Clearly, we may assume that $z_i \neq 0 (i \in \mathbb{N})$. Then one has $\alpha_i := 2\|z_i\| > 0 (i \in \mathbb{N})$, $\alpha := 2(\sum_i \|z_i\|^p)^{\frac{1}{p}} < \infty$ and $\sum_i (\frac{\alpha_i}{\alpha})^p = 1$. By 5.3, there are $d_i \in D$ with $\frac{1}{\alpha_i} z_i = \sigma'_D(d_i) (i \in \mathbb{N})$. Put $d := \sum_i \frac{\alpha_i}{\alpha} d_i \in D$. Then, due to 5.2(iii), [4], 4.5,



$$\|\sigma'_D(d) - \sum_{i=1}^n \frac{1}{\alpha} z_i\|^p = \|\sigma'_D\left(\sum_{i=n+1}^{\infty} \frac{\alpha_i}{\alpha} d_i\right)\|^p \leq \sum_{i=n+1}^{\infty} \left(\frac{\alpha_i}{\alpha}\right)^p$$

and thus $\sum_{i=1}^{\infty} z_i = \alpha \sigma'_D(d)$. Therefore, $S_p(D)$ is a p-Banach space. \square

Let $\sigma_D : D \rightarrow \widehat{\bigcirc}_p \circ S_p(D)$ denote σ'_D with the codomain restricted to $\widehat{\bigcirc}_p \circ S_p(D)$ ($D \in \mathbf{TC}_p$). Then one has

5.5 Lemma: For any $D \in \mathbf{TC}_p$ $\sigma_D : D \rightarrow \widehat{\bigcirc}_p \circ S_p(D)$ is a \mathbf{TC}_p -morphism.

Proof: Due to 5.2(i),(iii), [4], 4.5, one has for $\alpha \in \Omega_p$, $x_i \in D$ ($i \in \mathbb{N}$),

$$\|\sigma_D\left(\sum_i \alpha_i x_i\right) - \sum_{i=1}^n \alpha_i \sigma_D(x_i)\|^p = \|\sigma'_D\left(\sum_{i=n+1}^{\infty} \alpha_i x_i\right)\|^p \leq \sum_{i=n+1}^{\infty} |\alpha_i|^p ,$$

$n \in \mathbb{N}$, implying the assertion. \square

5.6 Theorem: The functor $\widehat{\bigcirc}_p : \mathbf{Ban}_p \rightarrow \mathbf{TC}_p$ has as a left adjoint the functor S_p with the object function $D \mapsto S_p(D)$. The unit of this adjunction is σ_D ($D \in \mathbf{TC}_p$).

Proof: Obviously, for a \mathbf{TC}_p -morphism $\psi : D \rightarrow \widehat{\bigcirc}_p(B)$ ($B \in \mathbf{Ban}_p$), a \mathbf{Ban}_p -morphism $\widehat{\psi} : S_p(D) \rightarrow B$ with $\psi = \widehat{\bigcirc}_p(\widehat{\psi}) \circ \sigma_D$ is uniquely determined. Each $z \in S_p(D)$ can be written as $z = \lambda \sigma_D(x)$ ($\lambda \in \mathbb{K}, x \in D$). If $\lambda \sigma_D(x) = \lambda' \sigma_D(x')$, then, for some $\gamma > \max\{|\lambda|, |\lambda'|\}$, $\frac{\lambda}{\gamma}x = \frac{\lambda'}{\gamma}x'$ holds. This implies $\lambda \psi(x) = \lambda' \psi(x')$, and therefore $\widehat{\psi} : S_p(D) \rightarrow B$, defined by $\widehat{\psi}(z) := \lambda \psi(x)$ for $z = \lambda \sigma_D(x)$ is well-defined. By a simple computation, $\widehat{\psi}$ is a \mathbf{Ban}_p -morphism and $\widehat{\psi}$ fulfills $\psi = \widehat{\bigcirc}_p(\widehat{\psi}) \circ \sigma_D$ thus finishing the proof. \square

For $D \in \mathbf{AC}_p$ define $S_p(D)$ just as in the case of $D \in \mathbf{TC}_p$; this is possible since only finite operations enter into the definition of $S_p(D)$. Put $N_p(D) := \{z \in S_p(D) \mid \|z\| = 0\}$. Since $\|-$ is a p-seminorm on $S_p(D)$, $N_p(D)$ is a subvector space of $S_p(D)$. Define $S_{p,fin}(D) := S_p(D)/N_p(D)$ and let $\pi_D : S_p(D) \rightarrow S_{p,fin}(D)$ be the canonical projection. Denote by $\sigma_{D,fin} : D \rightarrow \widehat{\bigcirc}_{p,fin} \circ S_{p,fin}(D)$ the restriction of $\pi_D \circ \sigma'_D$ to $\widehat{\bigcirc}_{p,fin} \circ S_{p,fin}(D)$. π_D is an isometry and one has the following three propositions:

5.7 Proposition: For $D \in \mathbf{AC}_p$ the following hold:

$$(i) \|\sigma'_D(x)\| \leq \|x\| (x \in D).$$

$$(ii) \text{For } x_i \in D (i \in \mathbb{N}), \alpha \in \Omega_{p,fin} \text{ one has } \sigma'_D(\sum_i \alpha_i x_i) = \sum_i \alpha_i \sigma'_D(x_i). \square$$

5.8 Proposition: For any p-absolutely convex space D , $\overset{\circ}{\bigcirc}_{p,fin}(S_{p,fin}(D)) \subset \sigma_{D,fin}(D) \subset \bigcirc_{p,fin}(S_{p,fin}(D))$ holds. \square

5.9 Proposition: For $D \in \mathbf{AC}_p$, $\sigma_{D,fin} : D \rightarrow \overset{\circ}{\bigcirc}_{p,fin}(S_{p,fin}(D))$ is an \mathbf{AC}_p -morphism. \square

5.10 Theorem: The functor $\overset{\circ}{\bigcirc}_{p,fin} : \mathbf{Vec}_p \rightarrow \mathbf{AC}_p$ has as a left adjoint the functor $S_{p,fin} : \mathbf{AC}_p \rightarrow \mathbf{Vec}_p$ with the object function $D \mapsto S_{p,fin}(D)$. The unit of this adjunction is $\sigma_{D,fin}$ ($D \in \mathbf{AC}_p$).

Proof: Let $\psi : D \rightarrow \overset{\circ}{\bigcirc}_{p,fin}(V)$ be an \mathbf{AC}_p -morphism, $V \in \mathbf{Vec}_p$. For $z \in S_p(D)$ there exist $\lambda \in \mathbb{K}, x \in D$ with $z = \lambda \sigma'_D(x)$, hence $\pi_D(z) = \lambda(\pi_D \circ \sigma'_D)(x) = \lambda \sigma_{D,fin}(x)$. Therefore, a \mathbf{Vec}_p -morphism $\hat{\psi} : S_{p,fin}(D) \rightarrow V$ with $\psi = \overset{\circ}{\bigcirc}_{p,fin}(\hat{\psi}) \circ \sigma_{D,fin}$ is uniquely determined. Define $\hat{\psi} : S_{p,fin}(D) \rightarrow V$ by $\hat{\psi}(\pi_D(z)) := \lambda \psi(x)$ for $z = \lambda \sigma'_D(x)$ ($\lambda \in \mathbb{K}, x \in D$). By the proof of 5.6, $\hat{\psi}$ is well-defined and \mathbb{K} -linear. For z as above one has $\|\hat{\psi}(\pi_D(z))\| = \|\lambda \psi(x)\| \leq |\lambda| \|x\| \leq |\lambda|$, hence $\|\hat{\psi}(\pi_D(z))\| \leq \|z\| = \|\pi_D(z)\|$. Thus $\hat{\psi}$ is a \mathbf{Vec}_p -morphism with $\psi = \overset{\circ}{\bigcirc}_{p,fin}(\hat{\psi}) \circ \sigma_{D,fin}$, finishing the proof. \square

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Positively Convex Spaces

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Introduction

In [7] Pumplün and Röhrl introduced the category **TC** (resp. \mathbf{TC}_{fin}) of (finitely) totally convex (t.c.) spaces, which are the Eilenberg-Moore algebras of the monad induced by the unit ball functor from the category of Banach spaces (resp. normed vector spaces) with linear contractions to the category of sets. We refer to [7] and [8] for all definitions and conventions, but in accordance with [2] we use the term “absolutely convex” for the spaces Pumplün and Röhrl call “finitely totally convex”. Later Pumplün [11] introduced the category **PC** (resp. \mathbf{PC}_{fin}) of (finitely) positively convex spaces and Pumplün and Röhrl ([10]) introduced the category **Conv** of convex spaces.

The regularly ordered Banach spaces (resp. regularly ordered normed vector spaces) ([11], 2.1) with positive linear contractions as morphisms form a category denoted by \mathbf{Ban}_1^+ (resp. \mathbf{Vec}_1^+). The positive part of the unit ball $\Delta_{fin}(V) := \{x \in V \mid \|x\| \leq 1 \text{ and } 0 \leq x\}, V \in \mathbf{Vec}_1^+, \Delta(B) := \Delta_{fin}(B), B \in \mathbf{Ban}_1^+$, induces a functor $\Delta : \mathbf{Ban}_1^+ \rightarrow \mathbf{Set}$ (resp. $\Delta_{fin} : \mathbf{Vec}_1^+ \rightarrow \mathbf{Set}$) having $l_1 : \mathbf{Set} \rightarrow \mathbf{Ban}_1^+, l_1(X) := \{f \in \mathbb{R}^X \mid \sum_{x \in X} |f(x)| < \infty\}, X \in \mathbf{Set}$ (resp. $l_{1,fin} : \mathbf{Set} \rightarrow \mathbf{Vec}_1^+, l_{1,fin}(X) := \mathbb{R}^{(X)}, X \in \mathbf{Set}$) as a left adjoint ([11], 2.2). Pumplün ([11], 2.9) proved, that the category **PC** (resp. \mathbf{PC}_{fin}) is the category of the Eilenberg-Moore algebras of the monad induced by this functor $\Delta : \mathbf{Ban}_1^+ \rightarrow \mathbf{Set}$ (resp. $\Delta_{fin} : \mathbf{Vec}_1^+ \rightarrow \mathbf{Set}$) with comparison functor $\hat{\Delta} : \mathbf{Ban}_1^+ \rightarrow \mathbf{PC}$ (resp. $\hat{\Delta}_{fin} : \mathbf{Vec}_1^+ \rightarrow \mathbf{PC}_{fin}$).

For categorial reasons ([13]), the functor $\hat{\Delta}$ (resp. $\hat{\Delta}_{fin}$) has a left adjoint. Several constructions of this left adjoint S of $\hat{\Delta}$ are known ([6], [11], [15]),

but these constructions give only little insight into the structure of $S(D)$ and how it depends on that of D since $S(D)$ ($D \in \mathbf{PC}$) is only constructed in several steps. In the present paper the left adjoints S of $\hat{\Delta}$ and S_{fin} of $\hat{\Delta}_{fin}$ are (in the case S up to completion) explicitly constructed in one step.

Furthermore, it was an open problem, how to give a characterization of separated (finitely) positively convex spaces D in terms of preseparated (finitely) positively convex spaces ([14], 4.1). A (finitely) positively convex space D is separated (in the sense of [15]) if and only if the family of \mathbf{PC} (resp. \mathbf{PC}_{fin})-morphisms $D \rightarrow \hat{\Delta}(\mathbb{R})$ (resp. $D \rightarrow \hat{\Delta}_{fin}(\mathbb{R})$) is point-separating ([15], 10.1, 11.3).

At first, we define separated (finitely) positively convex spaces by a (seemingly exotic) cancellation law, and later we see, that this definition coincides with the above notation of separated. Finally, as a consequence of [4], we prove that the category \mathbf{PC}_{psep} of preseparated positively convex spaces does not have a cogenerator. \square

§1 Separated Positively Convex Spaces

A totally convex (t.c.) space X is a non-empty set which admits every $\alpha \in \Omega := \{(\alpha_i)_{i \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} \mid \sum_i |\alpha_i| \leq 1\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, as \mathbb{N} -ary operation. The result of such an operation is written as a formal sum $\sum_i \alpha_i x_i$, $x_i \in X$ ($i \in \mathbb{N}$), and the operations are required to satisfy the following two axioms.

$$(\text{TC1}) \sum_i \delta_i^j y_i = y_j, \text{ for all } j \in \mathbb{N}, (y_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}.$$

$$(\text{TC2}) \sum_i \alpha_i (\sum_j \beta_j^i y_j) = \sum_j (\sum_i \alpha_i \beta_j^i) y_j, \text{ for all } \alpha, \beta^i \in \Omega, y_i \in X (i \in \mathbb{N}).$$

A morphism $f : X \rightarrow Y$ between t.c. spaces is a set mapping preserving these operations, i.e. $f(\sum_i \alpha_i x_i) = \sum_i \alpha_i f(x_i)$, $x_i \in X$ ($i \in \mathbb{N}$), $\alpha \in \Omega$ ([7], p. 963). We define $\Omega_{fin} := \{\alpha \in \Omega \mid |\text{supp } \alpha| < \infty\}$, where $\text{supp } \alpha := \{i \in \mathbb{N} \mid \alpha_i \neq 0\}$ for $\alpha \in \Omega$. An absolutely convex (a.c.) space is a non-empty Ω_{fin} -algebra satisfying (TC1) and the restriction of (TC2) to $\alpha, \beta^i \in \Omega_{fin}$ ($i \in \mathbb{N}$); \mathbf{AC} denotes the category of a.c. spaces and maps that preserve

the Ω_{fin} -operations ([7], p. 971). Furthermore ([2], §1), in case $\mathbb{K} = \mathbb{IR}$ we define $\Omega^+ := \{\alpha \in \Omega \mid \forall i \in \mathbb{N} \ \alpha_i \geq 0\}$, $\Omega_{fin}^+ := \Omega^+ \cap \Omega_{fin}$. Restricting the TC-operations (and thus the axioms and the conditions for morphisms) to Ω^+ (resp. Ω_{fin}^+) we get the category **PC** (resp. **PC_{fin}**) of (finitely) positively convex spaces ([11], 2.4). Similarly, putting $\Omega_{sc} := \{\alpha \in \Omega^+ \mid \sum_i \alpha_i = 1\}$ (resp. $\Omega_c := \Omega_{sc} \cap \Omega_{fin}$) and discarding the condition $X \neq \emptyset$ yields the category **SC** (resp. **Conv**) of superconvex (resp. convex) spaces ([10], p. 2). For **TC**, **AC**, **PC** and **PC_{fin}**, $0 := (0)_{i \in \mathbb{N}}$ is an algebra operation, which is constant on any space, hence 0 yields a nullary operation; we call its value the zero element of the space. \square

Let D be a finitely positively convex space. The so-called norm on D is defined by the Minkowski-functional $\|x\| := \inf\{\lambda \in [0, 1] \mid \text{there is } y \in D \text{ with } x = \lambda y\}$ ([11], 4.1). One should note that $\|\cdot\|$ is not a norm in the usual sense, because $\|x\| = 0$ in general does not imply $x = 0$ ([11], 3.3(iii)) nor is $\|\cdot\|$ positively homogeneous, i.e. it is not even a seminorm. Nevertheless we stick to the established name. \square

1.1 Remark: A. Wickenhäuser ([15], 9.2) (unpublished) for $D \in \mathbf{PC}$ defined a semi-metric d_w on D in the following way:

$$d_w((x, y)) := \inf\{\alpha \in [0, 2] \mid \text{there exist } r, s, t \in D \text{ and } n \in \mathbb{N}, n \geq 3 \text{ with } \frac{1}{n}y + \frac{1}{2}s = \frac{1}{n}x + \frac{\alpha}{n}r \text{ and } \frac{1}{n}x + \frac{1}{2}t = \frac{1}{n}y + \frac{\alpha}{n}r\} \quad (x, y \in D).$$

Obviously, the above set is non-empty (define $n := 4, \alpha := 2, r := \frac{1}{2}x + \frac{1}{2}y, s := x$ and $t := y$). A positively convex space D is separated i.e. for $x, y \in D$ with $x \neq y$ there exists a **PC**-morphism $\varphi : D \longrightarrow \hat{\Delta}(\mathbb{IR})$ with $\varphi(x) \neq \varphi(y)$ if and only if for all $x, y \in D$ $d_w((x, y)) = 0$ implies $x = y$ ([15], 10.1, 11.3). In the following we will proceed differently. \square

The following definition of separated (finitely) positively convex spaces seems to be very exotic, but it will turn out (3.6) that it is equivalent to the usual notion of separated.

1.2 Definition: (i) Let $D \in \mathbf{PC}(\mathbf{PC}_{fin})$. Then D is called separated if and only if for all $x, y \in D$ the following condition is fulfilled:

If, for every $\varepsilon \in]0, 1[$, there exist a $\beta \in]0, \frac{1}{2}]$ and $u, v \in D$ with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$ and $\|\frac{1}{2}u + \frac{1}{2}v\| < \varepsilon\beta$, then $x = y$ follows.

(ii) The full subcategory of $\mathbf{PC}(\mathbf{PC}_{fin})$ determined by the separated (finitely) positively convex spaces is denoted by $\mathbf{PC}_{sep}(\mathbf{PC}_{fin, sep})$. \square

Let $D \in \mathbf{PC}(\mathbf{PC}_{fin})$ and define a relation “ \sim ” on D by $x \sim y$ if and only if for every $\varepsilon \in]0, 1[$ there exist a $\beta \in]0, \frac{1}{2}]$ and $u, v \in D$ with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$ and $\|\frac{1}{2}u + \frac{1}{2}v\| < \varepsilon\beta$ ($x, y \in D$). Then we have the following

1.3 Theorem: For $D \in \mathbf{PC}(\mathbf{PC}_{fin})$ the above relation “ \sim ” is a congruence relation on D . The canonical projection $\pi_D : D \rightarrow D/\sim$ is the reflection mapping for the reflection from $\mathbf{PC}(\mathbf{PC}_{fin})$ into the subcategory $\mathbf{PC}_{sep}(\mathbf{PC}_{fin, sep})$, i.e. $\mathbf{PC}_{sep}(\mathbf{PC}_{fin, sep})$ is an ext-epi-reflective subcategory of $\mathbf{PC}(\mathbf{PC}_{fin})$.

Proof: Let $D \in \mathbf{PC}$. Obviously, the relation “ \sim ” is reflexive and symmetric. Let $x, y, z \in D$ with $x \sim y$ and $y \sim z$. For every ε with $0 < \varepsilon < 1$ there exist $\beta, \beta' \in]0, \frac{1}{2}]$, $u, v, u', v' \in D$ with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$, $\beta'y + \frac{1}{2}u' = \beta'z + \frac{1}{2}v'$ and $\|\frac{1}{2}u + \frac{1}{2}v\| < \frac{\varepsilon}{2}\beta$, $\|\frac{1}{2}u' + \frac{1}{2}v'\| < \frac{\varepsilon}{2}\beta'$. Without loss of generality we may assume $\beta = \beta'$. This implies

$$\frac{\beta}{2}x + \frac{1}{2}(\frac{1}{2}u + \frac{1}{2}u') = \frac{1}{2}(\beta x + \frac{1}{2}u) + \frac{1}{4}u' = \frac{1}{2}(\beta y + \frac{1}{2}v) + \frac{1}{4}u' = \frac{1}{2}(\beta y + \frac{1}{2}u') + \frac{1}{4}v = \frac{1}{2}(\beta z + \frac{1}{2}v') + \frac{1}{4}v = \frac{\beta}{2}z + \frac{1}{2}(\frac{1}{2}v + \frac{1}{2}v')$$

and by [11], 4.3

$\|\frac{1}{2}(\frac{1}{2}u + \frac{1}{2}u') + \frac{1}{2}(\frac{1}{2}v + \frac{1}{2}v')\| = \|\frac{1}{2}(\frac{1}{2}u + \frac{1}{2}v) + \frac{1}{2}(\frac{1}{2}u' + \frac{1}{2}v')\| \leq \frac{1}{2}\|\frac{1}{2}u + \frac{1}{2}v\| + \frac{1}{2}\|\frac{1}{2}u' + \frac{1}{2}v'\| < \frac{\varepsilon}{2}\beta$. Hence $x \sim z$, and “ \sim ” is transitive. Let $x_i, y_i \in D$ ($i \in \mathbb{N}$) and $\alpha \in \Omega^+$. Then, for every $\varepsilon > 0$, there exist $\beta_i \in]0, \frac{1}{2}]$, $u_i, v_i \in D$ with $\beta_i x_i + \frac{1}{2}u_i = \beta_i y_i + \frac{1}{2}v_i$ and $\|\frac{1}{2}u_i + \frac{1}{2}v_i\| < \frac{\varepsilon}{2}\beta_i$ ($i \in \mathbb{N}$). Without loss of generality let $(\beta_i)_{i \in \mathbb{N}}$ be a strictly decreasing zero sequence. There exists $m \in \mathbb{N}$ with $\sum_{i=m}^{\infty} \alpha_i < \frac{\varepsilon}{4}$. Define $\beta := \beta_m$. Then, for every $i \in \mathbb{N}$ with $i \geq m$, $\beta_i \leq \beta$ holds and $\beta < \beta_i$ otherwise. For $i \geq m$ one defines $u'_i := \frac{1}{2}u_i + (\beta - \beta_i)y_i$ and $v'_i := \frac{1}{2}v_i + (\beta - \beta_i)x_i$. This yields for every $i \geq m$

$$\begin{aligned}\frac{\beta}{2}x_i + \frac{1}{2}u'_i &= \frac{\beta_i}{2}x_i + \frac{\beta-\beta_i}{2}x_i + \frac{1}{2}(\frac{1}{2}u_i + (\beta - \beta_i)y_i) = \frac{1}{2}(\beta_i x_i + \frac{1}{2}u_i) + \frac{\beta-\beta_i}{2}x_i \\ + \frac{\beta-\beta_i}{2}y_i &= \frac{1}{2}(\beta_i y_i + \frac{1}{2}v_i) + \frac{\beta-\beta_i}{2}x_i + \frac{\beta-\beta_i}{2}y_i = \frac{\beta}{2}y_i + \frac{1}{2}(\frac{1}{2}v_i + (\beta - \beta_i)x_i) \\ &= \frac{\beta}{2}y_i + \frac{1}{2}v'_i \text{ and}\end{aligned}$$

$$\begin{aligned}\|\frac{1}{2}u'_i + \frac{1}{2}v'_i\| &= \|\frac{1}{2}(\frac{1}{2}u_i + (\beta - \beta_i)y_i) + \frac{1}{2}(\frac{1}{2}v_i + (\beta - \beta_i)x_i)\| \\ &= \|\frac{1}{2}(\frac{1}{2}u_i + \frac{1}{2}v_i) + (\beta - \beta_i)(\frac{1}{2}x_i + \frac{1}{2}y_i)\| \leq \frac{1}{2} \|\frac{1}{2}u_i + \frac{1}{2}v_i\| \\ &\quad + (\beta - \beta_i) \|\frac{1}{2}x_i + \frac{1}{2}y_i\| < \frac{\varepsilon}{4}\beta_i + (\beta - \beta_i) \leq \frac{\varepsilon}{4}\beta + (\beta - \beta_i).\end{aligned}$$

For each i with $1 \leq i \leq m-1$ let $u'_i := \frac{\beta}{2\beta_i}u_i, v'_i := \frac{\beta}{2\beta_i}v_i$. This leads to

$$\begin{aligned}\frac{\beta}{2}x_i + \frac{1}{2}u'_i &= \frac{\beta}{2}x_i + \frac{1}{2}(\frac{\beta}{2\beta_i}u_i) = \frac{\beta}{2\beta_i}(\beta_i x_i + \frac{1}{2}u_i) = \frac{\beta}{2\beta_i}(\beta_i y_i + \frac{1}{2}v_i) = \frac{\beta}{2}y_i + \frac{1}{2}(\frac{\beta}{2\beta_i}v_i) = \frac{\beta}{2}y_i + \frac{1}{2}v'_i \text{ and } \|\frac{1}{2}u'_i + \frac{1}{2}v'_i\| = \|\frac{1}{2}(\frac{\beta}{2\beta_i}u_i) + \frac{1}{2}(\frac{\beta}{2\beta_i}v_i)\| = \|\frac{\beta}{2\beta_i}(\frac{1}{2}u_i + \frac{1}{2}v_i)\| \leq \frac{\beta}{2\beta_i} \|\frac{1}{2}u_i + \frac{1}{2}v_i\| < \frac{\varepsilon}{4}\beta. \text{ This implies } \frac{\beta}{2} \sum_i \alpha_i x_i + \frac{1}{2} \sum_i \alpha_i u'_i = \sum_i \alpha_i (\frac{\beta}{2}x_i + \frac{1}{2}u'_i) = \sum_i \alpha_i (\frac{\beta}{2}y_i + \frac{1}{2}v'_i) = \frac{\beta}{2} \sum_i \alpha_i y_i + \frac{1}{2} \sum_i \alpha_i v'_i \text{ and } \|\frac{1}{2} \sum_i \alpha_i u'_i + \frac{1}{2} \sum_i \alpha_i v'_i\| = \|\sum_i \alpha_i (\frac{1}{2}u'_i + \frac{1}{2}v'_i)\| \leq \sum_i \alpha_i \|\frac{1}{2}u'_i + \frac{1}{2}v'_i\| = \sum_{i=1}^{m-1} \alpha_i \|\frac{1}{2}u'_i + \frac{1}{2}v'_i\| + \sum_{i=m}^{\infty} \alpha_i \|\frac{1}{2}u'_i + \frac{1}{2}v'_i\| \leq \sum_{i=1}^{m-1} \alpha_i (\frac{\varepsilon}{4}\beta) + \sum_{i=m}^{\infty} \alpha_i (\frac{\varepsilon}{4}\beta + (\beta - \beta_i)) < \varepsilon \frac{\beta}{2}. \text{ Thus } \sum_i \alpha_i x_i \sim \sum_i \alpha_i y_i \text{ follows, and "}" is a congruence relation on } D. \text{ Let } \pi : D \rightarrow D/\sim \text{ be the canonical projection. Let } x, y \in D \text{ be such that for every } \varepsilon \in]0, 1[\text{ there exist } \beta \in]0, \frac{1}{2}], u, v \in D, \text{ with } \pi(\beta x + \frac{1}{2}u) = \beta \pi(x) + \frac{1}{2}\pi(u) = \beta \pi(y) + \frac{1}{2}\pi(v) = \pi(\beta y + \frac{1}{2}v) \text{ and } \|\pi(\frac{1}{2}u + \frac{1}{2}v)\| = \|\frac{1}{2}\pi(u) + \frac{1}{2}\pi(v)\| < \frac{\varepsilon}{3}\beta.\end{aligned}$$

From $\beta x + \frac{1}{2}u \sim \beta y + \frac{1}{2}v$ we get elements $\sigma \in]0, \frac{1}{2}], u', v' \in D$ with $\sigma(\beta x + \frac{1}{2}u) + \frac{1}{2}u' = \sigma(\beta y + \frac{1}{2}v) + \frac{1}{2}v'$ and $\|\frac{1}{2}u' + \frac{1}{2}v'\| < (\frac{\varepsilon}{3}\beta)\sigma$. $\|\pi(\frac{1}{2}u + \frac{1}{2}v)\| < \frac{\varepsilon}{3}\beta$ implies the existence of an element $z \in D$ with $\pi(\frac{1}{2}u + \frac{1}{2}v) = \pi(z)$ and $\|z\| < \frac{\varepsilon}{3}\beta$ ([9], 1.5). From $\frac{1}{2}u + \frac{1}{2}v \sim z$ the existence of elements $\tau \in]0, \frac{1}{2}], u'', v'' \in D$ with $\tau(\frac{1}{2}u + \frac{1}{2}v) + \frac{1}{2}u'' = \tau z + \frac{1}{2}v''$ and $\|\frac{1}{2}u'' + \frac{1}{2}v''\| < (\frac{\varepsilon}{3}\beta)\tau$ follows. Put $u_0 := \sigma(\frac{\tau}{2}u + \frac{1}{2}u'') + \frac{\tau}{2}u'$ and $v_0 := \sigma(\frac{\tau}{2}v + \frac{1}{2}v'') + \frac{\tau}{2}v'$. Then $\frac{\sigma\tau\beta}{2}x + \frac{1}{2}u_0 = \frac{\sigma\tau\beta}{2}x + \frac{1}{2}(\sigma(\frac{\tau}{2}u + \frac{1}{2}u'') + \frac{\tau}{2}u') = \frac{\tau}{2}(\sigma(\beta x + \frac{1}{2}u) + \frac{1}{2}u') + \frac{1}{2}(\frac{\sigma}{2}u'') = \frac{\tau}{2}(\sigma(\beta y + \frac{1}{2}v) + \frac{1}{2}v') + \frac{1}{2}(\frac{\sigma}{2}u'') = \frac{\sigma\tau\beta}{2}y + \frac{1}{2}(\sigma(\frac{\tau}{2}v + \frac{1}{2}v'') + \frac{\tau}{2}v') = \frac{\sigma\tau\beta}{2}y + \frac{1}{2}v_0$ and

$$\|\frac{1}{2}u_0 + \frac{1}{2}v_0\| = \|\frac{1}{2}(\sigma(\frac{\tau}{2}u + \frac{1}{2}u'') + \frac{\tau}{2}u') + \frac{1}{2}(\sigma(\frac{\tau}{2}v + \frac{1}{2}v'') + \frac{\tau}{2}v')\|$$

$$\begin{aligned}
&= \left\| \frac{\sigma}{2} \left(\tau \left(\frac{1}{2}u + \frac{1}{2}v \right) + \frac{1}{2}u'' \right) + \frac{\tau}{2} \left(\frac{1}{2}u' + \frac{1}{2}v' \right) + \frac{\sigma}{2} \left(\frac{1}{2}u'' \right) \right\| \\
&= \left\| \frac{\sigma}{2} \left(\tau z + \frac{1}{2}v' \right) + \frac{\tau}{2} \left(\frac{1}{2}u' + \frac{1}{2}v' \right) + \frac{\sigma}{2} \left(\frac{1}{2}u'' \right) \right\| = \left\| \frac{\sigma\tau}{2}z + \frac{\sigma}{2} \left(\frac{1}{2}u'' + \frac{1}{2}v'' \right) + \frac{\tau}{2} \left(\frac{1}{2}u' + \frac{1}{2}v' \right) \right\| \\
&\leq \frac{\sigma\tau}{2} \|z\| + \frac{\sigma}{2} \left\| \frac{1}{2}u'' + \frac{1}{2}v'' \right\| + \frac{\tau}{2} \left\| \frac{1}{2}u' + \frac{1}{2}v' \right\| < \varepsilon \frac{\sigma\tau\beta}{2}. \text{ This implies } x \sim y, \text{ and } D/\sim \text{ is separated. Let } f : D \rightarrow E, E \in \mathbf{PC}_{sep}, \text{ be a PC-morphism. For } x, y \in D \text{ with } \pi(x) = \pi(y), \text{ for every } \varepsilon \in]0, 1[\text{ there exist a } \beta \in]0, \frac{1}{2}], u, v \in D \text{ with } \beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v \text{ and } \left\| \frac{1}{2}u + \frac{1}{2}v \right\| < \varepsilon\beta. \text{ This leads to } \beta f(x) + \frac{1}{2}f(u) = f(\beta x + \frac{1}{2}u) = f(\beta y + \frac{1}{2}v) = \beta f(y) + \frac{1}{2}f(v) \text{ and, by [11], 4.4(i), to } \left\| \frac{1}{2}f(u) + \frac{1}{2}f(v) \right\| = \left\| f \left(\frac{1}{2}u + \frac{1}{2}v \right) \right\| \leq \left\| \frac{1}{2}u + \frac{1}{2}v \right\| < \varepsilon\beta. \text{ Since } E \text{ is separated, we get } f(x) = f(y). \text{ Hence, there exists a unique PC-morphism } \psi : D/\sim \rightarrow E \text{ with } f = \psi \circ \pi_D. \text{ The proof in the finitary case is analogous. } \square \right.
\end{aligned}$$

A (finitely) positively convex space D is called preseparated if and only if for all $x, y \in D$, $\alpha \in]0, 1[$, $\alpha x = \alpha y$ implies $x = y$ ([14], 4.1). A convex space D is called preseparated if and only if for all $x, y, z \in D$, $\alpha \in]0, 1[$, $\alpha x + (1 - \alpha)z = \alpha y + (1 - \alpha)z$ implies $x = y$ ([10], 4.9). Now we have the following

1.4 Lemma: For $D \in \mathbf{PC}_{sep}$ ($\mathbf{PC}_{fin, sep}$) the following statements hold:

- (i) The underlying convex space is preseparated.
- (ii) D (as (finitely) positively convex space) is preseparated (c.f. [15], 10.8).
- (iii) For all $x, y \in D$, $\alpha \in]0, 1[$, $\alpha x + (1 - \alpha)y = 0$ implies $x = y = 0$.
- (iv) For all $x, y, z \in D$, α, β with $0 < \alpha, 0 \leq \beta$ and $\alpha + \beta \leq 1$, $\alpha x + \beta z = \alpha y + \beta z$ implies $x = y$.

Proof: (i) Let $x, y, z \in D$, $\alpha \in]0, 1[$ with $\alpha x + (1 - \alpha)z = \alpha y + (1 - \alpha)z$ and $0 < \varepsilon < 1$. For any $\frac{2}{2+\varepsilon} < \beta < 1$ we get ([2], 1.5), $\frac{1}{2}x + \frac{1}{2}(\frac{1-\beta}{\beta}z) = \frac{1}{2}\beta x + (1 - \beta)z = \frac{1}{2}\beta y + (1 - \beta)z = \frac{1}{2}y + \frac{1}{2}(\frac{1-\beta}{\beta}z)$. $\left\| \frac{1}{2}(\frac{1-\beta}{\beta}z) + \frac{1}{2}(\frac{1-\beta}{\beta}z) \right\| = \left\| \frac{1-\beta}{\beta}z \right\| \leq \frac{1-\beta}{\beta} < \frac{\varepsilon}{2}$ implies $x = y$, since D is separated.

(ii) This follows from 1.4(i).
(iii) Put $u := \frac{1-\alpha}{2}y$ and $v := \frac{\alpha}{2}x$. From $\frac{1}{2}(\frac{\alpha}{2}x) + \frac{1}{2}u = \frac{1}{2}(\frac{\alpha}{2}x) + \frac{1}{2}(\frac{1-\alpha}{2}y) = \frac{1}{2}(\frac{1-\alpha}{2}y) + \frac{1}{2}v$ and $\left\| \frac{1}{2}u + \frac{1}{2}v \right\| = \left\| \frac{1}{2}(\frac{1-\alpha}{2}y) + \frac{1}{2}(\frac{\alpha}{2}x) \right\| = \left\| \frac{1}{4}(\alpha x + (1 - \alpha)y) \right\| = 0$ we conclude $\frac{\alpha}{2}x = \frac{1-\alpha}{2}y$, since D is separated. This implies $\frac{\alpha}{2}x = \frac{1}{2}(\frac{\alpha}{2}x) +$

$\frac{1}{2}(\frac{\alpha}{2}x) = \frac{1}{2}(\frac{\alpha}{2}x) + \frac{1}{2}(\frac{1-\alpha}{2}y) = \frac{1}{4}(\alpha x + (1-\alpha)y) = 0 = \frac{\alpha}{2}0$. Now, from (ii) $x = 0$ follows and by a symmetric argument, $y = 0$.

(iv) We may assume $\alpha < 1$. From $\alpha x + (1-\alpha)(\frac{\beta}{1-\alpha}z) = \alpha x + \beta z = \alpha y + \beta z = \alpha y + (1-\alpha)(\frac{\beta}{1-\alpha}z)$, $x = y$ follows by 1.4(i). \square

§2 A Semi-Metric on Positively Convex Spaces

Let D be a (finitely) positively convex space. The mapping $d : D \times D \rightarrow \mathbb{R}$ is defined by $d((x, y)) := \inf\{\frac{\|\frac{1}{2}u + \frac{1}{2}v\|}{\beta} \mid \beta \in]0, \frac{1}{2}], u, v \in D \text{ and } \beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v\}$. Then one has $d((x, y)) \leq 2$ ($x, y \in D$) and the following

2.1 Proposition: Let $D \in \mathbf{PC}(\mathbf{PC}_{fin})$. Then $d : D \times D \rightarrow \mathbb{R}$ fulfills the following conditions: (i) $0 \leq d((x, y)) = d((y, x)) \quad (x, y \in D)$.

(ii) $d((x, z)) \leq d((x, y)) + d((y, z)) \quad (x, y, z \in D)$.

(iii) $d((\sum_i \alpha_i x_i, \sum_i \alpha_i y_i)) \leq \sum_i \alpha_i d((x_i, y_i)) \quad (x_i, y_i \in D \ (i \in \mathbb{N}), \alpha \in \Omega^+(\Omega_{fin}^+))$.

(iv) $d((\alpha x, \alpha y)) = \alpha d((x, y)) \quad (x, y \in D, 0 \leq \alpha \leq 1)$.

Proof: For all $x, y \in D$, $\frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}y + \frac{1}{2}x$ holds ([11], 2.5 (iv)). Therefore d is well-defined and (i) follows.

(ii) Let $\alpha, \gamma \in \mathbb{R}$ with $d((x, y)) < \alpha$ and $d((y, z)) < \gamma$. Then there exist $u, v, u_0, v_0 \in D$, $\beta, \beta_0 \in]0, \frac{1}{2}]$, with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$, $\beta_0 y + \frac{1}{2}u_0 = \beta_0 z + \frac{1}{2}v_0$ and $\frac{\|\frac{1}{2}u + \frac{1}{2}v\|}{\beta} < \alpha$, $\frac{\|\frac{1}{2}u_0 + \frac{1}{2}v_0\|}{\beta_0} < \gamma$. Without loss of generality we may assume $\beta = \beta_0$. Then one gets $\frac{\beta}{2}x + \frac{1}{2}(\frac{1}{2}u + \frac{1}{2}u_0) = \frac{1}{2}(\beta x + \frac{1}{2}u) + \frac{1}{4}u_0 = \frac{1}{2}(\beta y + \frac{1}{2}v) + \frac{1}{4}u_0 = \frac{1}{2}(\beta y + \frac{1}{2}u_0) + \frac{1}{4}v = \frac{1}{2}(\beta z + \frac{1}{2}v_0) + \frac{1}{4}v = \frac{\beta}{2}z + \frac{1}{2}(\frac{1}{2}v + \frac{1}{2}v_0)$ and $\frac{2}{\beta} \|\frac{1}{2}(\frac{1}{2}u + \frac{1}{2}u_0) + \frac{1}{2}(\frac{1}{2}v + \frac{1}{2}v_0)\| = \frac{2}{\beta} \|\frac{1}{2}(\frac{1}{2}u + \frac{1}{2}v) + \frac{1}{4}u_0 + \frac{1}{4}v_0\| \leq \frac{2}{\beta} (\frac{1}{2} \|\frac{1}{2}u + \frac{1}{2}v\| + \frac{1}{2} \|\frac{1}{2}u_0 + \frac{1}{2}v_0\|) < \alpha + \gamma$. This implies $d((x, z)) < \alpha + \gamma$ and hence $d((x, z)) \leq d((x, y)) + d((y, z))$.

(iii) Let $x_i, y_i \in D \in \mathbf{PC}$ ($i \in \mathbb{N}$), $\alpha \in \Omega^+$, and $\varepsilon > 0$. Then there exist $\beta_i \in]0, \frac{1}{2}]$, $u_i, v_i \in D$ with $\beta_i x_i + \frac{1}{2}u_i = \beta_i y_i + \frac{1}{2}v_i$ and $\frac{\|\frac{1}{2}u_i + \frac{1}{2}v_i\|}{\beta_i} \leq$

$d((x_i, y_i)) + \frac{\varepsilon}{2}$ ($i \in \mathbb{N}$). We may assume $(\beta_i)_{i \in \mathbb{N}}$ to be a strictly decreasing zero sequence. There exists $m \in \mathbb{N}$ with $\sum_{i=m}^{\infty} \alpha_i < \frac{\varepsilon}{4}$. Put $\beta := \beta_m$. Then, for all $i \in \mathbb{N}$ with $i \geq m$, $\beta_i \leq \beta$ holds and $\beta < \beta_i$ otherwise.

As in the proof of 1.3 for $i \geq m$ we define $u'_i = \frac{1}{2}u_i + (\beta - \beta_i)y_i$ and $v'_i := \frac{1}{2}v_i + (\beta - \beta_i)x_i$, and, for $1 \leq i \leq m-1$, $u'_i := \frac{\beta}{2\beta_i}u_i$ and $v'_i := \frac{\beta}{2\beta_i}v_i$. Then $\frac{\beta}{2}x_i + \frac{1}{2}u'_i = \frac{\beta}{2}y_i + \frac{1}{2}v'_i$ ($i \in \mathbb{N}$) follows. For $i \geq m$ $\|\frac{1}{2}u'_i + \frac{1}{2}v'_i\| \leq \frac{1}{2}\|\frac{1}{2}u_i + \frac{1}{2}v_i\| + (\beta - \beta_i)$ holds and, for $1 \leq i \leq m-1$, $\|\frac{1}{2}u'_i + \frac{1}{2}v'_i\| \leq \frac{\beta}{2\beta_i}\|\frac{1}{2}u_i + \frac{1}{2}v_i\|$. Furthermore, $\frac{\beta}{2}\sum_i \alpha_i x_i + \frac{1}{2}\sum_i \alpha_i u'_i = \frac{\beta}{2}\sum_i \alpha_i y_i + \frac{1}{2}\sum_i \alpha_i v'_i$ (see 1.3). Finally,

$$\begin{aligned} d\left(\left(\sum_i \alpha_i x_i, \sum_i \alpha_i y_i\right)\right) &\leq \frac{2}{\beta} \left\| \frac{1}{2} \sum_i \alpha_i u'_i + \frac{1}{2} \sum_i \alpha_i v'_i \right\| \\ &= \frac{2}{\beta} \left\| \sum_i \alpha_i \left(\frac{1}{2} u'_i + \frac{1}{2} v'_i \right) \right\| \leq \frac{2}{\beta} \sum_i \alpha_i \left\| \frac{1}{2} u'_i + \frac{1}{2} v'_i \right\| \\ &\leq \frac{2}{\beta} \left(\sum_{i=1}^{m-1} \alpha_i \left(\frac{\beta}{2\beta_i} \left\| \frac{1}{2} u_i + \frac{1}{2} v_i \right\| \right) + \sum_{i=m}^{\infty} \alpha_i \left(\frac{1}{2} \left\| \frac{1}{2} u_i + \frac{1}{2} v_i \right\| + (\beta - \beta_i) \right) \right) \end{aligned}$$

$\leq (\sum_{i=1}^{\infty} \alpha_i d((x_i, y_i))) + \varepsilon$ follows. The proof in the finitary case is analogous.

(iv) Let $x, y \in D$, $\alpha \in [0, 1]$ and $\gamma > 0$ with $d((\alpha x, \alpha y)) < \gamma$. We may assume $\alpha > 0$, otherwise the assertion is trivial. Then there exist $\beta \in [0, \frac{1}{2}]$, $u, v \in D$, with $\beta(\alpha x) + \frac{1}{2}u = \beta(\alpha y) + \frac{1}{2}v$ and $\frac{\|\frac{1}{2}u + \frac{1}{2}v\|}{\beta} < \gamma$. This implies $(\alpha\beta)x + \frac{1}{2}u = \beta(\alpha x) + \frac{1}{2}u = \beta(\alpha y) + \frac{1}{2}v = (\alpha\beta)y + \frac{1}{2}v$, and from $\frac{\|\frac{1}{2}u + \frac{1}{2}v\|}{\alpha\beta} < \frac{\gamma}{\alpha}$ we get $d((x, y)) < \frac{\gamma}{\alpha}$, thus $\alpha d((x, y)) < \gamma$. Consequently, $\alpha d((x, y)) \leq d((\alpha x, \alpha y))$ follows, and, by (iii), $d((\alpha x, \alpha y)) = \alpha d((x, y))$. \square

By 2.1(i), (ii), d is a semi-metric, and from 1.2 one gets

2.2 Proposition: Let $D \in \mathbf{PC}(\mathbf{PC}_{fin})$. Then D is separated if and only if d is a metric. (c.p. [15], 10.1, 11.3, with d replaced by d_w). \square

2.3 Lemma: For $D \in \mathbf{PC}(\mathbf{PC}_{fin})$, $x \in D$, $d((x, 0)) \leq \|x\|$ holds (c.p. [15], 9.3(iv), with d replaced by d_w).

Proof: Let $\alpha > 1$. From $\frac{1}{2(1+\alpha)}x + \frac{1}{2}(\frac{\alpha-1}{2(1+\alpha)}x) = \frac{1}{2(1+\alpha)}0 + \frac{1}{2}(\frac{1}{2}x)$ and $2(1+\alpha)\|\frac{1}{2}(\frac{\alpha-1}{2(1+\alpha)}x) + \frac{1}{2}(\frac{1}{2}x)\| = 2(1+\alpha)\|\frac{\alpha}{2(1+\alpha)}x\| \leq \alpha\|x\|$ one gets $d((x, 0)) \leq \alpha\|x\|$, thus $d((x, 0)) \leq \|x\|$. \square

2.4 Remark: Put $\mathbb{R}^+ := \{x \in \mathbb{R} | x \geq 0\}$ and let V be a real normed vector space with norm $\|-|\$. A cone C in V is a subset $C \subset V$ with $C+C \subset C$ and $\mathbb{R}^+C \subset C$. C is called proper if and only if $C \cap (-C) = \{0\}$.

A proper cone C induces an order on V by defining $x \leq y$ if and only if $y - x \in C$ ($x, y \in V$); this order is reflexive, antisymmetric, transitive and compatible with addition and \mathbb{R}^+ -scalar-multiplication on V . Conversely, any such order is induced by a proper cone, namely $C = \{x \in V | x \geq 0\}$.

$(V, \|-|)$ is called regular ordered if and only if V is ordered by a closed proper cone C and the norm $\|-|$ is a Riesz norm with respect to C that is

- 1) For all $x \in V, y \in C$ with $-y \leq x \leq y$ $\|x\| \leq \|y\|$ holds.
- 2) For every $x \in V, \varepsilon > 0$ there exists a $y \in C$ with $-y \leq x \leq y$ and $\|y\| \leq \|x\| + \varepsilon$.

Obviously, this is equivalent to $\|x\| = \inf\{\|y\| | -y \leq x \leq y\}$ for all $x \in V$. A regular ordered Banach space is a complete regular ordered vector space. The subcategory of Vec_1^+ , determined by all regular ordered real normed vector spaces and positive linear contractions, is denoted by Vec_1^+ . Ban_1^+ is the full subcategory of Vec_1^+ of regular ordered Banach spaces. For a regular ordered normed vector space V (resp. regular ordered Banach space V), the positive part of its unit ball $\Delta_{fin}(V) := \{x \in V | \|x\| \leq 1 \text{ and } x \geq 0\}$ induces a functor $\Delta_{fin} : \text{Vec}_1^+ \rightarrow \text{Set}$ (resp. $\Delta : \text{Ban}_1^+ \rightarrow \text{Set}$). $\Delta_{fin}(V)$ (resp. $\Delta(B), B \in \text{Ban}_1^+$) is in a canonical way a (finitely) positively convex space. This induces a full and faithful functor $\hat{\Delta}_{fin} : \text{Vec}_1^+ \rightarrow \text{PC}_{fin}$ (resp. $\hat{\Delta} : \text{Ban}_1^+ \rightarrow \text{PC}$) ([11], p. 89). The postulate that C is closed in a regularly ordered normed vector space is not a real restriction, because one has the well known

2.5 Lemma: Let V be a real normed vector space with Riesz norm $\|-|$

respect to a proper cone C in V . Then $(V, \|\cdot\|)$ is regular ordered by the closure \overline{C} of C with respect to $\|\cdot\|$. \square

2.6 Lemma (c.p. [15], 9.6, with d replaced by d_w): (i) Let $f : C \rightarrow D$ be a PC_{fin} -morphism and $x, y \in C$. Then $d((f(x), f(y))) \leq d((x, y))$ holds.

(ii) Let $(V, \|\cdot\|)$ be a regularly ordered normed vector space and $x, y \in \hat{\Delta}_{fin}(V)$. Then the metric $d : \hat{\Delta}_{fin}(V) \times \hat{\Delta}_{fin}(V) \rightarrow \mathbb{R}$ is $d((x, y)) = \|x - y\|$.

Proof: (i) For all $\beta \in]0, \frac{1}{2}]$, $u, v \in D$, $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$ implies $\beta f(x) + \frac{1}{2}f(u) = f(\beta x + \frac{1}{2}u) = f(\beta y + \frac{1}{2}v) = \beta f(y) + \frac{1}{2}f(v)$, hence $d((f(x), f(y))) \leq d((x, y))$.

(ii) For $x, y \in \hat{\Delta}_{fin}(V)$, $\|x - y\| = \inf\{\|z\| \mid z \leq x - y \leq z\}$ holds. Thus there exist $z, c_0, c_1 \in C$ with $x + z = y + c_1$ and $y + z = x + c_0$. This leads to $z = \frac{1}{2}c_0 + \frac{1}{2}c_1$ and $\frac{1}{2}x + \frac{1}{2}(\frac{1}{2}c_0) = \frac{1}{2}y + \frac{1}{2}(\frac{1}{2}c_1)$. From $2\|\frac{1}{2}(\frac{1}{2}c_0) + \frac{1}{2}(\frac{1}{2}c_1)\| = 2\|\frac{1}{2}(\frac{1}{2}c_0 + \frac{1}{2}c_1)\| = 2\|\frac{1}{2}z\| \leq \|z\|$ we get $d((x, y)) \leq \|z\|$, hence $d((x, y)) \leq \|x - y\|$ follows. Conversely, let $\beta \in]0, \frac{1}{2}]$, $u, v \in \hat{\Delta}_{fin}(V)$ with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$. This implies $-(\frac{1}{2}u + \frac{1}{2}v) \leq \beta(x - y) \leq \frac{1}{2}u + \frac{1}{2}v$, thus $\beta\|x - y\| = \|\beta(x - y)\| \leq \|\frac{1}{2}u + \frac{1}{2}v\|$ and from $\|x - y\| \leq \frac{\|\frac{1}{2}u + \frac{1}{2}v\|}{\beta}$ one concludes $\|x - y\| \leq d((x, y))$. \square

2.7 Proposition: For all $D \in \text{PC}_{fin}$ $d = d_w$ holds.

Proof: Let $x, y \in D$, $\beta \in]0, \frac{1}{2}]$, $u, v \in D$ with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$ and $\gamma > \frac{\|\frac{1}{2}u + \frac{1}{2}v\|}{\beta}$. Let $n \in \mathbb{N}$ with $\frac{1}{n} < \min\{\frac{\beta}{2}, \frac{1}{\gamma}, \frac{2}{1+\gamma}\}$ and define $u' := \frac{1}{n\beta}u$ and $v' := \frac{1}{n\beta}v$. Then one has

$$\frac{1}{2n}x + \frac{1}{4}u' = \frac{1}{2n\beta}(\beta x + \frac{1}{2}u) = \frac{1}{2n\beta}(\beta y + \frac{1}{2}v) = \frac{1}{2n}y + \frac{1}{4}v'.$$

As $\|\frac{1}{2}u' + \frac{1}{2}v'\| = \|\frac{1}{2}(\frac{1}{n\beta}u) + \frac{1}{2}(\frac{1}{n\beta}v)\| \leq \frac{1}{n\beta}\|\frac{1}{2}u + \frac{1}{2}v\| < \frac{\gamma}{n}$ there exists an element $z \in D$ with $\frac{1}{2}u' + \frac{1}{2}v' = \frac{\gamma}{n}z$. As $\frac{1}{2}(\frac{1}{2n}x + \frac{\gamma}{2n}z) = \frac{1}{2}(\frac{1}{2n}x + \frac{1}{2}(\frac{1}{2}u' + \frac{1}{2}v')) = \frac{1}{2}(\frac{1}{2n}x + \frac{1}{4}u') + \frac{1}{2}(\frac{1}{4}v') = \frac{1}{2}(\frac{1}{2n}y + \frac{1}{4}v') + \frac{1}{2}(\frac{1}{4}v') = \frac{1}{2}(\frac{1}{2n}y + \frac{1}{2}v')$ and $\|\frac{1}{2n}x + \frac{\gamma}{2n}z\| \leq \frac{1}{2n}\|x\| + \frac{\gamma}{2n}\|z\| < \frac{1}{2n} + \frac{1}{2} < 1$, $\|\frac{1}{2n}y + \frac{1}{2}v'\| < 1$ we get $\frac{1}{2n}x + \frac{\gamma}{2n}z = \frac{1}{2n}y + \frac{1}{2}v'$.

([3], 1.1). In the same way $\frac{1}{2n}y + \frac{\gamma}{2n}z = \frac{1}{2n}x + \frac{1}{2}u'$ follows, which implies $d_w((x, y)) \leq \gamma$ and $d_w((x, y)) \leq d((x, y))$.

Conversely, let $r, s, t \in D$, $n \geq 3$, $\alpha \in [0, 2]$ with $\frac{1}{n}x + \frac{\alpha}{n}r = \frac{1}{n}y + \frac{1}{2}s$ and $\frac{1}{n}y + \frac{\alpha}{n}r = \frac{1}{n}x + \frac{1}{2}t$. We may assume $n \geq 2\alpha$. Let $0 < \varepsilon < 1$, $0 < \beta < \frac{\varepsilon}{1+\varepsilon}$ and define $s' := (1 - \beta)(\frac{1}{2}s) + \frac{\beta}{2}(\frac{1}{n}x + \frac{1}{n}y)$, $t' := (1 - \beta)(\frac{1}{2}t) + \frac{\beta}{2}(\frac{1}{n}x + \frac{1}{n}y)$. As $\frac{1}{2}(\frac{1}{n}x + \frac{1}{n}y) + \frac{1}{2}(\frac{2\alpha}{n}r) = \frac{1}{2}(\frac{1}{n}x + \frac{\alpha}{n}r) + \frac{1}{2}(\frac{1}{n}y + \frac{\alpha}{n}r) = \frac{1}{2}(\frac{1}{n}y + \frac{1}{2}s) + \frac{1}{2}(\frac{1}{n}x + \frac{1}{2}t) = \frac{1}{2}(\frac{1}{n}x + \frac{1}{n}y) + \frac{1}{2}(\frac{1}{2}s + \frac{1}{2}t)$ holds,

(*) $\beta(\frac{1}{n}x + \frac{1}{n}y) + (1 - \beta)(\frac{2\alpha}{n}r) = \beta(\frac{1}{n}x + \frac{1}{n}y) + (1 - \beta)(\frac{1}{2}s + \frac{1}{2}t)$ follows ([2], 1.5). We have $\frac{1-\beta}{n}x + \frac{1}{2}t' = \frac{1-\beta}{n}x + \frac{1}{2}((1 - \beta)(\frac{1}{2}t) + \frac{\beta}{2}(\frac{1}{n}x + \frac{1}{n}y)) = \frac{1-\beta}{2}(\frac{1}{n}x + \frac{1}{2}t) + \frac{1-\beta}{2n}x + \frac{\beta}{4}(\frac{1}{n}x + \frac{1}{n}y) = \frac{1-\beta}{2}(\frac{1}{n}y + \frac{\alpha}{n}r) + \frac{1-\beta}{2n}x + \frac{\beta}{4}(\frac{1}{n}x + \frac{1}{n}y) = \frac{1-\beta}{2}(\frac{1}{n}x + \frac{\alpha}{n}r) + \frac{1-\beta}{2}(\frac{1}{n}y) + \frac{\beta}{4}(\frac{1}{n}x + \frac{1}{n}y) = \frac{1-\beta}{2}(\frac{1}{n}y + \frac{1}{2}s) + \frac{1-\beta}{2}(\frac{1}{n}y) + \frac{\beta}{4}(\frac{1}{n}x + \frac{1}{n}y) = \frac{1-\beta}{n}y + \frac{1}{2}s'$, and (*) yields

$$\begin{aligned} \frac{n}{1-\beta} \left\| \frac{1}{2}t' + \frac{1}{2}s' \right\| &\leq \frac{n}{2(1-\beta)} \left\| \beta\left(\frac{1}{n}x + \frac{1}{n}y\right) + (1 - \beta)\left(\frac{1}{2}s + \frac{1}{2}t\right) \right\| = \\ \frac{n}{2(1-\beta)} \left\| \beta\left(\frac{1}{n}x + \frac{1}{n}y\right) + (1 - \beta)\left(\frac{2\alpha}{n}r\right) \right\| &\leq \frac{n}{2(1-\beta)} (\beta \left\| \frac{1}{n}x + \frac{1}{n}y \right\| \\ + (1 - \beta) \left\| \frac{2\alpha}{n}r \right\|) &\leq \frac{n}{2(1-\beta)} \left(\frac{2\beta}{n} + \frac{2\alpha(1-\beta)}{n} \right) = \alpha + \frac{\beta}{1-\beta} < \alpha + \varepsilon. \end{aligned}$$

This implies $d((x, y)) \leq \alpha$ and $d((x, y)) \leq d_w((x, y))$, hence $d = d_w$. \square

§3 The Left Adjoint of the Comparison Functor

U_{fin} (resp. U) the canonical forgetful functor from \mathbf{PC}_{fin} (resp. \mathbf{PC}) to \mathbf{Set} , is the Eilenberg-Moore category of $\Delta_{fin} : \mathbf{Vec}_1^+ \rightarrow \mathbf{Set}$ (resp. $\Delta : \mathbf{Ban}_1^+ \rightarrow \mathbf{Set}$). The comparison functor is $\hat{\Delta}_{fin} : \mathbf{Vec}_1^+ \rightarrow \mathbf{PC}_{fin}$ (resp. $\hat{\Delta} : \mathbf{Ban}_1^+ \rightarrow \mathbf{PC}$) ([11], 2.9 (the assertion in the finitary case can be proved as in the infinitary case)).

One knows from general theorems ([13]) that the functors $\hat{\Delta}$ and $\hat{\Delta}_{fin}$ have a left adjoint $S : \mathbf{PC} \rightarrow \mathbf{Ban}_1^+$ (resp. $S_{fin} : \mathbf{PC}_{fin} \rightarrow \mathbf{Vec}_1^+$). A

construction of S in several steps can be found in [11], 4.21, 4.22. In [15], 11.1, there is a construction of S in three steps. The following construction of S (3.5) is a generalization of the construction of the left adjoint of the functor $\hat{\Delta}_{sep} : \mathbf{Ban}_1^+ \rightarrow \mathbf{PC}_{sep}$, i.e. the restriction of $\hat{\Delta} : \mathbf{Ban}_1^+ \rightarrow \mathbf{PC}$ to \mathbf{PC}_{sep} , in [15], 11.1. \square

Let $(V, \| - \|)$ be a regularly ordered vector space with cone C . Then the completion $(\hat{V}, \| - \|)$ of $(V, \| - \|)$ is a regularly ordered Banach space respect to the proper cone \bar{C} , \bar{C} the closure of C in $(\hat{V}, \| - \|)$. The dense embedding $j_V : V \hookrightarrow \hat{V}$ is the reflection mapping of the reflection $V \mapsto \hat{V}$, showing \mathbf{Ban}_1^+ to be a dense-reflective subcategory of \mathbf{Vec}_1^+ (cp. [16]). \square

Let $D \in \mathbf{PC}$ (resp. $D \in \mathbf{PC}_{fin}$). A relation “ \approx ” on $\mathbb{R}^+ \times D \times D$ is defined as follows: $(\alpha, x_1, x_2) \sim (\beta, y_1, y_2)$ if and only if there exists a $\lambda > \alpha + \beta$, such that for every $\varepsilon > 0$ there are a $\sigma \in]0, \frac{1}{2}]$ and $u, v \in D$ with

$$\sigma\left(\frac{\alpha}{\lambda}x_1 + \frac{\beta}{\lambda}y_2\right) + \frac{1}{2}u = \sigma\left(\frac{\alpha}{\lambda}x_2 + \frac{\beta}{\lambda}y_1\right) + \frac{1}{2}v \text{ and } \left\| \frac{1}{2}u + \frac{1}{2}v \right\| < \varepsilon\sigma.$$

Let $\pi : D \rightarrow D/\sim$ be the reflection mapping from \mathbf{PC} into \mathbf{PC}_{sep} (resp. from \mathbf{PC}_{fin} into $\mathbf{PC}_{fin, sep}$) (see 1.3). Then $(\alpha, x_1, x_2) \approx (\beta, y_1, y_2)$ if and only if there exists a $\lambda > \alpha + \beta$ with $\pi\left(\frac{\alpha}{\lambda}x_1 + \frac{\beta}{\lambda}y_2\right) = \pi\left(\frac{\alpha}{\lambda}x_2 + \frac{\beta}{\lambda}y_1\right)$. Then one has

3.1 Lemma: The above relation “ \approx ” is an equivalence relation.

Proof: For $(\alpha, x_1, x_2) \in \mathbb{R}^+ \times D \times D$ put $\lambda := 2\alpha$, and “ \approx ” is shown to be reflexive. Obviously, it is symmetric. Let $(\alpha, x_1, x_2) \approx (\beta, y_1, y_2)$ and $(\beta, y_1, y_2) \approx (\gamma, z_1, z_2)$. Thus there exist $\lambda > \alpha + \beta$, $\lambda' > \beta + \gamma$ with $\pi\left(\frac{\alpha}{\lambda}x_1 + \frac{\beta}{\lambda}y_2\right) = \pi\left(\frac{\alpha}{\lambda}x_2 + \frac{\beta}{\lambda}y_1\right)$ and $\pi\left(\frac{\beta}{\lambda'}y_1 + \frac{\gamma}{\lambda'}z_2\right) = \pi\left(\frac{\beta}{\lambda'}y_2 + \frac{\gamma}{\lambda'}z_1\right)$. We may assume $\lambda = \lambda' \geq \max\{2\beta, \alpha + \beta, \alpha + \gamma, \beta + \gamma\}$. This implies $\frac{1}{2}\left(\frac{\beta}{\lambda}\pi(y_1) + \frac{\beta}{\lambda}\pi(y_2)\right) + \frac{1}{2}\left(\frac{\alpha}{\lambda}\pi(x_1) + \frac{\gamma}{\lambda}\pi(z_2)\right) = \frac{1}{2}\left(\frac{\alpha}{\lambda}\pi(x_1) + \frac{\beta}{\lambda}\pi(y_2)\right) + \frac{1}{2}\left(\frac{\beta}{\lambda}\pi(y_1) + \frac{\gamma}{\lambda}\pi(z_2)\right) = \frac{1}{2}\left(\frac{\beta}{\lambda}\pi(y_1) + \frac{\beta}{\lambda}\pi(y_2)\right) + \frac{1}{2}\left(\frac{\alpha}{\lambda}\pi(x_2) + \frac{\gamma}{\lambda}\pi(z_1)\right) = \frac{1}{2}\left(\frac{\beta}{\lambda}\pi(y_1) + \frac{\beta}{\lambda}\pi(y_2)\right) + \frac{1}{2}\left(\frac{\alpha}{\lambda}\pi(x_2) + \frac{\gamma}{\lambda}\pi(z_1)\right)$. Since D/\sim is preseparated, we conclude $\pi\left(\frac{\alpha}{\lambda}x_1 + \frac{\gamma}{\lambda}z_2\right) = \frac{\alpha}{\lambda}\pi(x_1) + \frac{\gamma}{\lambda}\pi(z_2) =$

$$\frac{\alpha}{\lambda}\pi(x_2) + \frac{\gamma}{\lambda}\pi(z_1) = \pi\left(\frac{\alpha}{\lambda}x_2 + \frac{\gamma}{\lambda}z_1\right), \text{ thus } (\alpha, x_1, x_2) \approx (\gamma, z_1, z_2). \quad \square$$

Let $E := (\mathbb{R}^+ \times D \times D)/\approx$. In the following we write $\overline{(\alpha, x_1, x_2)}$ for the \approx -equivalence class of (α, x_1, x_2) in E . A \mathbb{R} -scalar multiplication and an addition on E is defined by $\mu\overline{(\alpha, x_1, x_2)} := \overline{(\mu\alpha, x_1, x_2)}$ for $\mu \geq 0$ and $\mu\overline{(\alpha, x_1, x_2)} := \overline{(-\mu\alpha, x_2, x_1)}$ for $\mu < 0$ and

$$\overline{(\alpha, x_1, x_2)} + \overline{(\beta, y_1, y_2)} := \overline{\left(\gamma, \frac{\alpha}{\gamma}x_1 + \frac{\beta}{\gamma}y_1, \frac{\alpha}{\gamma}x_2 + \frac{\beta}{\gamma}y_2\right)} \quad (\gamma > \alpha + \beta).$$

3.2 Lemma: With the above definitions, E is a real vector space.

Proof: Obviously, the scalar multiplication is well-defined. Let $\overline{(\alpha, x_1, x_2)} = \overline{(\alpha', x'_1, x'_2)}$ and $\overline{(\beta, y_1, y_2)} = \overline{(\beta', y'_1, y'_2)}$. Then there exists a $\gamma > \max\{\alpha + \beta, \alpha' + \beta'\}$ with

$$\begin{aligned} \overline{(\alpha, x_1, x_2)} + \overline{(\beta, y_1, y_2)} &= \overline{\left(\gamma, \frac{\alpha}{\gamma}x_1 + \frac{\beta}{\gamma}y_1, \frac{\alpha}{\gamma}x_2 + \frac{\beta}{\gamma}y_2\right)} \text{ and} \\ \overline{(\alpha', x'_1, x'_2)} + \overline{(\beta', y'_1, y'_2)} &= \overline{\left(\gamma, \frac{\alpha'}{\gamma}x'_1 + \frac{\beta'}{\gamma}y'_1, \frac{\alpha'}{\gamma}x'_2 + \frac{\beta'}{\gamma}y'_2\right)}. \end{aligned}$$

Furthermore, there exists a $\lambda > \max\{\alpha + \alpha', \beta + \beta', \alpha + \beta, \alpha' + \beta'\}$ with $\pi\left(\frac{\alpha}{\lambda}x_1 + \frac{\alpha'}{\lambda}x'_2\right) = \pi\left(\frac{\alpha}{\lambda}x_2 + \frac{\alpha'}{\lambda}x'_1\right)$ and $\pi\left(\frac{\beta}{\lambda}y_1 + \frac{\beta'}{\lambda}y'_2\right) = \pi\left(\frac{\beta}{\lambda}y_2 + \frac{\beta'}{\lambda}y'_1\right)$. This yields $\pi\left(\frac{\gamma}{2\gamma}\left(\frac{\alpha}{\lambda}x_1 + \frac{\beta}{\lambda}y_1\right) + \frac{\gamma}{2\gamma}\left(\frac{\alpha'}{\lambda}x'_2 + \frac{\beta'}{\lambda}y'_2\right)\right) = \frac{1}{2}\pi\left(\frac{\alpha}{\lambda}x_1 + \frac{\alpha'}{\lambda}x'_2\right) + \frac{1}{2}\pi\left(\frac{\beta}{\lambda}y_1 + \frac{\beta'}{\lambda}y'_2\right) = \frac{1}{2}\pi\left(\frac{\alpha}{\lambda}x_2 + \frac{\alpha'}{\lambda}x'_1\right) + \frac{1}{2}\pi\left(\frac{\beta}{\lambda}y_2 + \frac{\beta'}{\lambda}y'_1\right) = \pi\left(\frac{\gamma}{2\gamma}\left(\frac{\alpha}{\lambda}x_2 + \frac{\beta}{\lambda}y_2\right) + \frac{\gamma}{2\gamma}\left(\frac{\alpha'}{\lambda}x'_1 + \frac{\beta'}{\lambda}y'_1\right)\right)$, thus $\overline{(\alpha, x_1, x_2)} + \overline{(\beta, y_1, y_2)} = \overline{(\alpha', x'_1, x'_2)} + \overline{(\beta', y'_1, y'_2)}$. Hence, the addition is well-defined as well. Now, one proves the vector space axioms by a straightforward computation. \square

We define a mapping $\|\cdot\| : E \rightarrow \mathbb{R}$ by $\|(\alpha, x_1, x_2)\| := \alpha d(\pi(x_1), \pi(x_2))$, where d is the semi-metric defined in 2.1, π the reflection mapping from 1.3 and we have the following

3.3 Lemma: $(E, \|\cdot\|)$ is a real normed vector space.

Proof: At first we show that $\|\cdot\|$ is well-defined. Let $\overline{(\alpha, x_1, x_2)} = \overline{(\gamma, y_1, y_2)}$ and $\beta \in]0, \frac{1}{2}]$, $u, v \in D$ with $\beta\pi(x_1) + \frac{1}{2}\pi(u) = \beta\pi(x_2) + \frac{1}{2}\pi(v)$. Then there exists a $\lambda > \alpha + \gamma$ with $\pi(\frac{\alpha}{\lambda}x_1 + \frac{\gamma}{\lambda}y_2) = \pi(\frac{\alpha}{\lambda}x_2 + \frac{\gamma}{\lambda}y_1)$. Put $x := \frac{1}{2}x_1 + \frac{1}{2}x_2$ and $\lambda_0 := \lambda - \alpha\beta > 0$. First let $\gamma > 0$. Then the following operations are well defined and $\frac{\alpha\beta}{\lambda}\pi(x) + (1 - \frac{\alpha\beta}{\lambda})(\frac{\gamma\beta}{2\lambda_0}\pi(y_1) + \frac{\alpha}{4\lambda_0}\pi(u)) = \frac{\alpha}{2\lambda}(\beta\pi(x_1) + \frac{1}{2}\pi(u)) + \frac{\beta}{2}\pi(\frac{\alpha}{\lambda}x_2 + \frac{\gamma}{\lambda}y_1) = \frac{\alpha}{2\lambda}(\beta\pi(x_2) + \frac{1}{2}\pi(v)) + \frac{\beta}{2}\pi(\frac{\alpha}{\lambda}x_1 + \frac{\gamma}{\lambda}y_2) = \frac{\alpha\beta}{\lambda}\pi(x) + (1 - \frac{\alpha\beta}{\lambda})(\frac{\gamma\beta}{2\lambda_0}\pi(y_2) + \frac{\alpha}{4\lambda_0}\pi(v))$. From 1.4(i), $\frac{\gamma\beta}{2\lambda_0}\pi(y_1) + \frac{1}{2}(\frac{\alpha}{2\lambda_0}\pi(u)) = \frac{\gamma\beta}{2\lambda_0}\pi(y_2) + \frac{1}{2}(\frac{\alpha}{2\lambda_0}\pi(v))$, and we have $\frac{2\lambda_0}{\gamma\beta}\|\frac{1}{2}(\frac{\alpha}{2\lambda_0}\pi(u)) + \frac{1}{2}(\frac{\alpha}{2\lambda_0}\pi(v))\| \leq \frac{\alpha}{\gamma}\frac{\|\frac{1}{2}\pi(u) + \frac{1}{2}\pi(v)\|}{\beta}$. This implies $d((\pi(y_1), \pi(y_2))) \leq \frac{\alpha}{\gamma}d((\pi(x_1), \pi(x_2)))$ and therefore one gets $\|\overline{(\gamma, y_1, y_2)}\| = \gamma d((\pi(y_1), \pi(y_2))) \leq \alpha d((\pi(x_1), \pi(x_2))) = \|\overline{(\alpha, x_1, x_2)}\|$. Obviously, for $\gamma = 0$ this holds, too, and, by symmetry, $\|\overline{(\gamma, y_1, y_2)}\| = \|\overline{(\alpha, x_1, x_2)}\|$. Thus $\|\cdot\|$ is well-defined. Obviously, $\|\cdot\|$ is homogenous. Next, let $(\alpha, x_1, x_2), (\gamma, y_1, y_2) \in E$ and $\beta_0, \beta_1 \in]0, \frac{1}{2}]$, $a_0, b_0, a_1, b_1 \in D$ with $\beta_0\pi(x_1) + \frac{1}{2}\pi(a_0) = \beta_0\pi(x_2) + \frac{1}{2}\pi(b_0)$ and $\beta_1\pi(y_1) + \frac{1}{2}\pi(a_1) = \beta_1\pi(y_2) + \frac{1}{2}\pi(b_1)$. We assume $\beta_0 \leq \beta_1$ and define $a_2 := \frac{\beta_0}{\beta_1}a_1$ and $b_2 := \frac{\beta_0}{\beta_1}b_1$. Then $\beta_0\pi(y_1) + \frac{1}{2}\pi(a_2) = \frac{\beta_0}{\beta_1}(\beta_1\pi(y_1) + \frac{1}{2}\pi(a_1)) = \frac{\beta_0}{\beta_1}(\beta_1\pi(y_2) + \frac{1}{2}\pi(b_1)) = \beta_0\pi(y_2) + \frac{1}{2}\pi(b_2)$. Let $\lambda > \alpha + \gamma$ with

$$\overline{(\alpha, x_1, x_2)} + \overline{(\gamma, y_1, y_2)} = \overline{(\lambda, \frac{\alpha}{\lambda}x_1 + \frac{\gamma}{\lambda}y_1, \frac{\alpha}{\lambda}x_2 + \frac{\gamma}{\lambda}y_2)}.$$

Now, put $a_3 := \frac{\alpha}{2\lambda}a_0 + \frac{\gamma}{2\lambda}a_2$ and $b_3 := \frac{\alpha}{2\lambda}b_0 + \frac{\gamma}{2\lambda}b_2$. Then $\frac{\beta_0}{2}(\frac{\alpha}{\lambda}\pi(x_1) + \frac{\gamma}{\lambda}\pi(y_1)) + \frac{1}{2}\pi(a_3) = \frac{\alpha}{2\lambda}(\beta_0\pi(x_1) + \frac{1}{2}\pi(a_0)) + \frac{\gamma}{2\lambda}(\beta_0\pi(y_1) + \frac{1}{2}\pi(a_2)) = \frac{\alpha}{2\lambda}(\beta_0\pi(x_2) + \frac{1}{2}\pi(b_0)) + \frac{\gamma}{2\lambda}(\beta_0\pi(y_2) + \frac{1}{2}\pi(b_2)) = \frac{\beta_0}{2}(\frac{\alpha}{\lambda}\pi(x_2) + \frac{\gamma}{\lambda}\pi(y_2)) + \frac{1}{2}\pi(b_3)$, follows and

$$\begin{aligned} & \frac{2}{\beta_0}\left\|\frac{1}{2}\pi(a_3) + \frac{1}{2}\pi(b_3)\right\| \\ &= \frac{2}{\beta_0}\left\|\frac{\alpha}{2\lambda}(\frac{1}{2}\pi(a_0) + \frac{1}{2}\pi(b_0)) + \frac{\gamma}{2\lambda}(\frac{\beta_0}{\beta_1}(\frac{1}{2}\pi(a_1) + \frac{1}{2}\pi(b_1)))\right\| \\ &\leq \frac{1}{\lambda}\left(\alpha\frac{\|\frac{1}{2}\pi(a_0) + \frac{1}{2}\pi(b_0)\|}{\beta_0} + \gamma\frac{\|\frac{1}{2}\pi(a_1) + \frac{1}{2}\pi(b_1)\|}{\beta_1}\right). \end{aligned}$$

This implies

$$\|\overline{(\alpha, x_1, x_2)} + \overline{(\gamma, y_1, y_2)}\| = \|\overline{(\lambda, \frac{\alpha}{\lambda}x_1 + \frac{\gamma}{\lambda}y_1, \frac{\alpha}{\lambda}x_2 + \frac{\gamma}{\lambda}y_2)}\|$$

$= \lambda d((\frac{\alpha}{\lambda}\pi(x_1) + \frac{\gamma}{\lambda}\pi(y_1), \frac{\alpha}{\lambda}\pi(x_2) + \frac{\gamma}{\lambda}\pi(y_2))) \leq \alpha d((\pi(x_1), \pi(x_2))) +$
 $\gamma d((\pi(y_1), \pi(y_2))) = \|\overline{(\alpha, x_1, x_2)}\| + \|\overline{(\gamma, y_1, y_2)}\|.$ Finally, let $\overline{(\alpha, x_1, x_2)} \in E$ with $\alpha d((\pi(x_1), \pi(x_2))) = \|\overline{(\alpha, x_1, x_2)}\| = 0$, i.e. $\alpha = 0$ or $d((\pi(x_1), \pi(x_2))) = 0$. In case $\alpha = 0$, $\overline{(\alpha, x_1, x_2)} = \overline{(0, 0, 0)}$ holds, and $d((\pi(x_1), \pi(x_2))) = 0$ implies $\pi(x_1) = \pi(x_2)$ by 2.2. This is equivalent with $\overline{(1, x_1, x_2)} = \overline{(0, 0, 0)}$, thus $\overline{(\alpha, x_1, x_2)} = \overline{(0, 0, 0)}$, and $(E, \|\cdot\|)$ is a real normed vector space. \square

3.4 Proposition: $C := \{\overline{(\alpha, x, y)} \in E \mid y = 0\}$ is a proper cone in E and E is a regularly ordered normed vector space with proper cone \overline{C} (see 2.5).

Proof: Obviously, $C + C \subset C$ and $\mathbb{R}^+C \subset C$ hold. Let $\overline{(\alpha, x, 0)}, \overline{(\beta, y, 0)} \in C$ with $\overline{(\alpha, x, 0)} = -\overline{(\beta, y, 0)} = \overline{(\beta, 0, y)}$. Then there exists a $\lambda > \alpha + \beta$ with $\frac{\alpha}{\lambda}\pi(x) + \frac{\lambda-\alpha}{\lambda}(\frac{\beta}{\lambda-\alpha}\pi(y)) = \frac{\alpha}{\lambda}\pi(x) + \frac{\beta}{\lambda}\pi(y) = \pi(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y) = \pi(0) = 0$. $\alpha = 0$ implies $\overline{(\alpha, x, 0)} = \overline{(0, 0, 0)}$. For $0 < \alpha$ we get $\pi(x) = 0$ from 1.4(iii). $\pi(x) = 0 = \pi(0)$ means $\overline{(\alpha, x, 0)} = \overline{(0, 0, 0)}$. Consequently, $C \cap (-C) = \{\overline{(0, 0, 0)}\}$ holds, and C is a proper cone in E .

Let $\overline{(\alpha, x_1, x_2)} \in E$, $\overline{(\beta, y_1, 0)} \in C$ with $-(\overline{(\beta, y_1, 0)}) \leq \overline{(\alpha, x_1, x_2)} \leq (\overline{(\beta, y_1, 0)})$. We show that $\|\overline{(\alpha, x_1, x_2)}\| \leq \|\overline{(\beta, y_1, 0)}\|$ holds. Without loss of generality we may assume $\alpha = \beta = 1$. There exist elements $\overline{(\alpha', x', 0)}, \overline{(\beta', y', 0)} \in C$ with $\overline{(1, 0, y_1)} + \overline{(\alpha', x', 0)} = \overline{(1, x_1, x_2)}$ and $\overline{(1, x_1, x_2)} + \overline{(\beta', y', 0)} = \overline{(1, y_1, 0)}$. Thus there is a $\gamma > \max\{1 + \alpha', 1 + \beta'\}$ with

$$\overline{(\gamma, \frac{\alpha'}{\gamma}x', \frac{1}{\gamma}y_1)} = \overline{(1, x_1, x_2)} \text{ and } \overline{(\gamma, \frac{1}{\gamma}x_1 + \frac{\beta'}{\gamma}y', \frac{1}{\gamma}x_2)} = \overline{(1, y_1, 0)}.$$

This implies the existence of an element $\lambda > \max\{2, \gamma + 1, \alpha' + \beta'\}$ with $\frac{1}{\lambda}\pi(x_1) + \frac{1}{\lambda}\pi(y_1) = \pi(\frac{\gamma}{\lambda}(\frac{1}{\gamma}y_1) + \frac{1}{\lambda}x_1) = \pi(\frac{\gamma}{\lambda}(\frac{\alpha'}{\gamma}x') + \frac{1}{\lambda}x_2) = \frac{1}{\lambda}\pi(x_2) + \frac{\alpha'}{\lambda}\pi(x')$ and $\frac{1}{\lambda}\pi(x_2) + \frac{1}{\lambda}\pi(y_1) = \pi(\frac{\gamma}{\lambda}(\frac{1}{\gamma}x_2) + \frac{1}{\lambda}y_1) = \pi(\frac{\gamma}{\lambda}(\frac{1}{\gamma}x_1 + \frac{\beta'}{\gamma}y') + \frac{1}{\lambda}0) = \frac{1}{\lambda}\pi(x_1) + \frac{\beta'}{\lambda}\pi(y')$. Let $\sigma \in [0, \frac{1}{2}]$, $u, v \in D$ with $\sigma\pi(y_1) + \frac{1}{2}\pi(u) = \frac{1}{2}\pi(v)$. From the above equations, one gets

$$\frac{1}{\lambda}(\frac{1}{2}\pi(x_1) + \frac{1}{2}\pi(x_2)) + \frac{1}{2}(\frac{2}{\lambda}\pi(y_1)) = \frac{1}{2}(\frac{1}{\lambda}\pi(x_1) + \frac{1}{\lambda}\pi(y_1))$$

$$\begin{aligned}
& + \frac{1}{2} \left(\frac{1}{\lambda} \pi(x_2) + \frac{1}{\lambda} \pi(y_1) \right) = \frac{1}{2} \left(\frac{1}{\lambda} \pi(x_2) + \frac{\alpha'}{\lambda} \pi(x') \right) + \frac{1}{2} \left(\frac{1}{\lambda} \pi(x_1) + \frac{\beta'}{\lambda} \pi(y') \right) \\
& = \frac{1}{\lambda} \left(\frac{1}{2} \pi(x_1) + \frac{1}{2} \pi(x_2) \right) + \frac{1}{2} \left(\frac{\alpha'}{\lambda} \pi(x') + \frac{\beta'}{\lambda} \pi(y') \right). \text{ 1.4(iv) implies } \frac{2}{\lambda} \pi(y_1) = \frac{\alpha'}{\lambda} \pi(x') + \frac{\beta'}{\lambda} \pi(y'). \text{ From } \frac{1}{2\lambda} \pi(y_1) + \frac{1}{2} \left(\frac{2}{\lambda} \pi(x_1) + \frac{\beta'}{\lambda} \pi(y') \right) = \frac{1}{2} \left(\frac{1}{\lambda} \pi(x_1) + \frac{1}{\lambda} \pi(y_1) \right) + \frac{1}{2} \left(\frac{1}{\lambda} \pi(x_1) + \frac{\beta'}{\lambda} \pi(y') \right) = \frac{1}{2} \left(\frac{1}{\lambda} \pi(x_2) + \frac{\alpha'}{\lambda} \pi(x') \right) + \frac{1}{2} \left(\frac{1}{\lambda} \pi(x_2) + \frac{1}{\lambda} \pi(y_1) \right) = \frac{1}{2\lambda} \pi(y_1) + \frac{1}{2} \left(\frac{2}{\lambda} \pi(x_2) + \frac{\alpha'}{\lambda} \pi(x') \right) \text{ we get } \frac{2}{\lambda} \pi(x_1) + \frac{\beta'}{\lambda} \pi(y') = \frac{2}{\lambda} \pi(x_2) + \frac{\alpha'}{\lambda} \pi(x') \text{ (1.4(iv)). Put } u' := \frac{\beta'\sigma}{2\lambda} y' + \frac{1}{2\lambda} u \text{ and } v' := \frac{\alpha'\sigma}{2\lambda} x' + \frac{1}{2\lambda} u. \text{ Then we conclude } \frac{\sigma}{2\lambda} \pi(x_1) + \frac{1}{2} \pi(u') = \frac{1}{2} \left(\frac{\sigma}{2} \left(\frac{2}{\lambda} \pi(x_1) + \frac{\beta'}{\lambda} \pi(y') \right) \right) + \frac{1}{4\lambda} \pi(u) = \frac{1}{2} \left(\frac{\sigma}{2} \left(\frac{2}{\lambda} \pi(x_2) + \frac{\alpha'}{\lambda} \pi(x') \right) \right) + \frac{1}{4\lambda} \pi(u) = \frac{\sigma}{2\lambda} \pi(x_2) + \frac{1}{2} \pi(v') \text{ and}
\end{aligned}$$

$$\begin{aligned}
& \frac{2\lambda}{\sigma} \left\| \frac{1}{2} \pi(u') + \frac{1}{2} \pi(v') \right\| = \frac{2\lambda}{\sigma} \left\| \frac{\sigma}{4} \left(\frac{\alpha'}{\lambda} \pi(x') + \frac{\beta'}{\lambda} \pi(y') \right) + \frac{1}{2\lambda} \pi(u) \right\| \\
& = \frac{2\lambda}{\sigma} \left\| \frac{\sigma}{4} \left(\frac{2}{\lambda} \pi(y_1) \right) + \frac{1}{2\lambda} \pi(u) \right\| = \frac{2\lambda}{\sigma} \left\| \frac{1}{2\lambda} (\sigma \pi(y_1) + \frac{1}{2} \pi(u)) + \frac{1}{4\lambda} \pi(u) \right\| \\
& = \frac{2\lambda}{\sigma} \left\| \frac{1}{2\lambda} \left(\frac{1}{2} \pi(v) \right) + \frac{1}{4\lambda} \pi(u) \right\| \leq \frac{\left\| \frac{1}{2} \pi(u) + \frac{1}{2} \pi(v) \right\|}{\sigma}.
\end{aligned}$$

Hence, $\|\overline{(1, x_1, x_2)}\| = d((\pi(x_1), \pi(x_2))) \leq d((\pi(y_1), \pi(0))) = \|\overline{(1, y_1, 0)}\|.$ Now, let $\overline{(\alpha, x_1, x_2)} \in E$ and $\varepsilon > 0.$ Let $\sigma \in]0, \frac{1}{2}]$, $u, v \in D$ with $\sigma \pi(x_1) + \frac{1}{2} \pi(u) = \sigma \pi(x_2) + \frac{1}{2} \pi(v)$ and for $\alpha > 0$, $\frac{\|\frac{1}{2}\pi(u) + \frac{1}{2}\pi(v)\|}{\sigma} \leq d((\pi(x_1), \pi(x_2))) + \frac{\varepsilon}{\alpha}.$ Put $w := \frac{1}{2}u + \frac{1}{2}v$ and $b := \overline{(\frac{\alpha}{\sigma}, w, 0)} \in C.$ We will prove $\overline{(\alpha, x_1, x_2)} + \overline{(\frac{\alpha}{\sigma}, w, 0)} = \overline{(\frac{\alpha}{\sigma}, v, 0)}$ and $\overline{(\frac{\alpha}{\sigma}, w, 0)} + \overline{(\alpha, x_2, x_1)} = \overline{(\frac{\alpha}{\sigma}, u, 0)}.$ For this, let $\gamma > \alpha + \frac{\alpha}{\sigma}$ with $\overline{(\alpha, x_1, x_2)} + \overline{(\frac{\alpha}{\sigma}, w, 0)} = \overline{(\gamma, \frac{\alpha}{\gamma}x_1 + \frac{\alpha}{\sigma\gamma}w, \frac{\alpha}{\gamma}x_2)}$ and $\overline{(\frac{\alpha}{\sigma}, w, 0)} + \overline{(\alpha, x_2, x_1)} = \overline{(\gamma, \frac{\alpha}{\sigma\gamma}w + \frac{\alpha}{\gamma}x_2, \frac{\alpha}{\gamma}x_1)}.$ Let $\lambda > \gamma + \frac{\alpha}{\sigma}.$ Then one has $\sigma \pi(\frac{\alpha}{\lambda}x_1 + \frac{\alpha}{\sigma\lambda}w) = \frac{\alpha}{\lambda}(\sigma \pi(x_1) + \frac{1}{2}\pi(u)) + \frac{\alpha}{2\lambda}\pi(v) = \frac{\alpha}{\lambda}(\sigma \pi(x_2) + \frac{1}{2}\pi(v)) + \frac{\alpha}{2\lambda}\pi(v) = \sigma \pi(\frac{\alpha}{\lambda}x_2 + \frac{\alpha}{\sigma\lambda}v).$ From 1.4(ii), one gets $\pi(\frac{\gamma}{\lambda}(\frac{\alpha}{\gamma}x_1 + \frac{\alpha}{\sigma\gamma}w)) = \pi(\frac{\alpha}{\lambda}x_1 + \frac{\alpha}{\sigma\lambda}w) = \pi(\frac{\alpha}{\lambda}x_2 + \frac{\alpha}{\sigma\lambda}v) = \pi(\frac{\gamma}{\lambda}(\frac{\alpha}{\gamma}x_2 + \frac{\alpha}{\sigma\lambda}v)),$ and this implies the first assertion. From $\sigma \pi(\frac{\alpha}{\sigma\lambda}w + \frac{\alpha}{\lambda}x_2) = \frac{\alpha}{\lambda}(\sigma \pi(x_2) + \frac{1}{2}\pi(v)) + \frac{\alpha}{2\lambda}\pi(u) = \frac{\alpha}{\lambda}(\sigma \pi(x_1) + \frac{1}{2}\pi(u)) + \frac{\alpha}{2\lambda}\pi(u) = \sigma \pi(\frac{\alpha}{\lambda}x_1 + \frac{\alpha}{\sigma\lambda}u)$ one derives as above $\pi(\frac{\gamma}{\lambda}(\frac{\alpha}{\gamma}x_1 + \frac{\alpha}{\sigma\lambda}u)) = \pi(\frac{\alpha}{\lambda}x_1 + \frac{\alpha}{\sigma\lambda}u) = \pi(\frac{\alpha}{\sigma\lambda}w + \frac{\alpha}{\lambda}x_2) = \pi(\frac{\gamma}{\lambda}(\frac{\alpha}{\sigma\gamma}w + \frac{\alpha}{\gamma}x_2)),$ and this leads to the second assertion. Hence, $-b \leq \overline{(\alpha, x_1, x_2)} \leq b$ holds. In case $\alpha = 0$ we have $b = \overline{(\frac{\alpha}{\sigma}, w, 0)} =$

$\overline{(0, 0, 0)} = \overline{(\alpha, x_1, x_2)}$, thus $\|b\| \leq \|(\alpha, x_1, x_2)\| + \varepsilon$. Now, by 2.3, in case $\alpha > 0$ we conclude

$$\begin{aligned} \|b\| &= \frac{\alpha}{\sigma} \|\overline{(1, w, 0)}\| = \frac{\alpha}{\sigma} d((\pi(w), 0)) \leq \alpha \frac{\|\frac{1}{2}\pi(u) + \frac{1}{2}\pi(v)\|}{\sigma} \\ &\leq \alpha d((\pi(x_1), \pi(x_2))) + \varepsilon = \|\overline{(\alpha, x_1, x_2)}\| + \varepsilon. \text{ Thus, } E \text{ is regularly ordered by } \overline{C}. \end{aligned}$$

For $D \in \mathbf{PC}_{fin}$ (resp. $D \in \mathbf{PC}$) we define $S_{fin}(D) := (E, \|-|)$ (resp. $S(D) := (\hat{E}, \|-|)$), where \hat{E} is the completion of E in \mathbf{Ban}_1^+). The mappings $\sigma'_{D,fin} : D \rightarrow S_{fin}(D)$ and $\sigma'_D : D \rightarrow S(D)$ are defined by $\sigma'_{D,fin}(x) := \overline{(1, x, 0)}$ ($x \in D \in \mathbf{PC}_{fin}$), $\sigma'_D(x) := in \circ \sigma'_{D,fin}(x)$ ($x \in D \in \mathbf{PC}$), where $in : S_{fin}(D) \hookrightarrow S(D)$ is the isometric inclusion.

The mappings $\sigma_D : D \rightarrow \hat{\Delta}(S(D))$ ($D \in \mathbf{PC}$) (resp. $\sigma_{D,fin} : D \rightarrow \hat{\Delta}_{fin}(S_{fin}(D))$ ($D \in \mathbf{PC}_{fin}$)) are defined by $\sigma_D(x) := \sigma'_D(x)$ ($x \in D$) (resp. $\sigma_{D,fin}(x) := \sigma'_{D,fin}(x)$ ($x \in D$)). Then one has

3.5 Theorem: S (resp. S_{fin}) extends canonically to a functor $S : \mathbf{PC} \rightarrow \mathbf{Ban}_1^+$ (resp. $S_{fin} : \mathbf{PC}_{fin} \rightarrow \mathbf{Vec}_1^+$), which is left adjoint to $\hat{\Delta} : \mathbf{Ban}_1^+ \rightarrow \mathbf{PC}$ (resp. $\hat{\Delta}_{fin} : \mathbf{Vec}_1^+ \rightarrow \mathbf{PC}_{fin}$). The unit of this adjunction is σ_D ($D \in \mathbf{PC}$) (resp. $\sigma_{D,fin}$ ($D \in \mathbf{PC}_{fin}$))).

Proof: By 2.3, σ_D ($D \in \mathbf{PC}$) and $\sigma_{D,fin}$ ($D \in \mathbf{PC}_{fin}$) are well-defined. Obviously, for $D \in \mathbf{PC}_{fin}$, $\sigma_{D,fin} : D \rightarrow S_{fin}(D)$ is a \mathbf{PC}_{fin} -morphism. Let $D \in \mathbf{PC}$, $x_i \in D$ ($i \in \mathbb{N}$), $\alpha \in \Omega^+$. For all $n \in \mathbb{N}$ one has

$$\begin{aligned} \sum_{i=1}^n \alpha_i \sigma_D(x_i) + \overline{(1, \sum_{i=n+1}^{\infty} \alpha_i x_i, 0)} &= \overline{(1, \sum_{i=1}^n \alpha_i x_i, 0)} + \overline{(1, \sum_{i=n+1}^{\infty} \alpha_i x_i, 0)} \\ &= \overline{(1, \sum_{i=1}^{\infty} \alpha_i x_i, 0)} = \sigma_D(\sum_i \alpha_i x_i). \text{ By 2.3, this leads to} \\ \|\sigma_D(\sum_i \alpha_i x_i) - \sum_{i=1}^n \alpha_i \sigma_D(x_i)\| &= \left\| \overline{(1, \sum_{i=n+1}^{\infty} \alpha_i x_i, 0)} \right\| \leq \sum_{i=n+1}^{\infty} \alpha_i, \end{aligned}$$

which implies $\sigma_D(\sum_i \alpha_i x_i) = \sum_i \alpha_i \sigma_D(x_i)$, i.e. σ_D is a \mathbf{PC} -morphism.

Let $D \in \mathbf{PC}_{fin}$ and $f : D \rightarrow \hat{\Delta}_{fin}(V)$, $V \in \mathbf{Vec}_1^+$, a \mathbf{PC}_{fin} -morphism. For all $x_1, x_2 \in D$, $\sigma_{D,fin}(x_1) - \sigma_{D,fin}(x_2) = \overline{(1, x_1, 0)} - \overline{(1, x_2, 0)} = \overline{(1, x_1, x_2)}$ holds. Consequently, a \mathbf{Vec}_1^+ -morphism $\varphi : S_{fin}(D) \rightarrow V$ with $\hat{\Delta}_{fin}(\varphi) \circ \sigma_{D,fin} = f$ is uniquely determined if it exists. Define $\varphi : S_{fin}(D) \rightarrow V$ by $\varphi(\overline{(\alpha, x_1, x_2)}) := \alpha(f(x_1) - f(x_2))$ ($\overline{(\alpha, x_1, x_2)} \in S_{fin}(D)$). Let $\overline{(\alpha, x_1, x_2)} = (\beta, y_1, y_2)$. Thus there exists a $\lambda > \alpha + \beta$ with $\pi(\frac{\alpha}{\lambda}x_1 + \frac{\beta}{\lambda}y_2) = \pi(\frac{\alpha}{\lambda}x_2 + \frac{\beta}{\lambda}y_1)$. By 1.3, there is a \mathbf{PC}_{fin} -morphism $\psi : D/\sim \rightarrow \hat{\Delta}_{fin}(V)$ with $\psi \circ \pi = f$. Thus we get $\frac{\alpha}{\lambda}f(x_1) + \frac{\beta}{\lambda}f(y_2) = \psi \circ \pi(\frac{\alpha}{\lambda}x_1 + \frac{\beta}{\lambda}y_2) = \psi \circ \pi(\frac{\alpha}{\lambda}x_2 + \frac{\beta}{\lambda}y_1) = \frac{\alpha}{\lambda}f(x_2) + \frac{\beta}{\lambda}f(y_1)$. This implies $\alpha(f(x_1) - f(x_2)) = \beta(f(y_1) - f(y_2))$, and the above mapping $\varphi : S_{fin}(D) \rightarrow V$ is well-defined.

Now, it is easily verified that φ is \mathbb{IR} -linear. For $\overline{(\alpha, x_1, x_2)} \in S_{fin}(D)$, 2.6 (i), (ii) yields $\|\varphi(\overline{(\alpha, x_1, x_2)})\| = \|\alpha(f(x_1) - f(x_2))\| = \alpha\|f(x_1) - f(x_2)\| = ad((f(x_1), f(x_2)) = ad((\psi \circ \pi(x_1), \psi \circ \pi(x_2))) \leq ad((\pi(x_1), \pi(x_2))) =$

$\|\overline{(\alpha, x_1, x_2)}\|$. For every $\overline{(\alpha, x, 0)} \in C$, $\varphi(\overline{(\alpha, x, 0)}) = \alpha f(x) \geq 0$ holds, i.e. φ is a positive linear contraction. Obviously, $\hat{\Delta}_{fin}(\varphi) \circ \sigma_{D,fin} = f$ is fulfilled, and even our assertion is proved in the finitary case. The proof in the infinitary case consists simply of a completion. \square

3.6 Theorem (c.f. [15], 11.3): Let $D \in \mathbf{PC}$ (resp. $D \in \mathbf{PC}_{fin}$). Then the following statements are equivalent: (i) D is separated.

- (ii) If, for $x, y \in D$, for every $\varepsilon \in]0, 1[$ there exist a $\beta \in]0, \frac{1}{2}]$, $u, v \in D$ with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$ and $\|\frac{1}{2}u + \frac{1}{2}v\| < \varepsilon\beta$, then $x = y$ holds.
- (iii) For all $x, y \in D$, $d((x, y)) = 0$ if and only if $x = y$ holds.
- (iv) σ_D (resp. $\sigma_{D,fin}$) is injective.
- (v) For all $x, y \in D$ with $x \neq y$ there exists a \mathbf{PC} -morphism $f : D \rightarrow \hat{\Delta}(\mathbb{IR})$ (resp. a \mathbf{PC}_{fin} -morphism $f : D \rightarrow \hat{\Delta}_{fin}(\mathbb{IR})$) with $f(x) \neq f(y)$.

Proof: (i) \iff (ii): Because of definition 1.2.

(ii) \implies (iii): This follows from 2.2, 2.7.

(iii) \implies (iv): For all $x, y \in D$, $\sigma_D(x) = \sigma_D(y)$ (resp. $\sigma_{D,fin}(x) = \sigma_{D,fin}(y)$)

if and only if $\overline{(1, x, 0)} = \overline{(1, y, 0)}$, if and only if $\pi(x) = \pi(y)$ holds, which implies the assertion (see 1.3).

(iv) \Rightarrow (v): Let $D \in \mathbf{PC}$ and $\sigma_D : D \rightarrow \hat{\Delta}(S(D))$ injective. Let $x, y \in D$ with $x \neq y$. Then we have $\sigma'_D(x) \neq \sigma'_D(y)$, and by [12], 9.2, there exists a positive linear contraction $\varphi : S(D) \rightarrow \hat{\Delta}(\mathbb{R})$ with $\varphi(\sigma'_D(x)) \neq \varphi(\sigma'_D(y))$. The mapping $f := \varphi \circ \sigma'_D : D \rightarrow \hat{\Delta}(\mathbb{R})$ is a \mathbf{PC} -morphism with $f(x) \neq f(y)$. The proof for $D \in \mathbf{PC}_{fin}$ is analogous.

(v) \Rightarrow (ii): Let $D \in \mathbf{PC}$ and $x, y \in D$ be such that for every ϵ with $0 < \epsilon < 1$ there exist $\beta \in]0, \frac{1}{2}]$, $u, v \in D$, with $\beta x + \frac{1}{2}u = \beta y + \frac{1}{2}v$ and $\|\frac{1}{2}u + \frac{1}{2}v\| < \epsilon\beta$. If $f : D \rightarrow \hat{\Delta}(\mathbb{R})$ is a \mathbf{PC} -morphism, we get $\beta f(x) + \frac{1}{2}f(u) = f(\beta x + \frac{1}{2}u) = f(\beta y + \frac{1}{2}v) = \beta f(y) + \frac{1}{2}f(v)$ and $\|\frac{1}{2}f(u) + \frac{1}{2}f(v)\| = \|f(\frac{1}{2}u + \frac{1}{2}v)\| \leq \|\frac{1}{2}u + \frac{1}{2}v\| < \epsilon\beta$. This implies $f(x) = f(y)$, since $\hat{\Delta}(\mathbb{R})$ is separated. By our assumption, this leads to $x = y$. The proof for $D \in \mathbf{PC}_{fin}$ is analogous. \square

It is surprising that separated \mathbf{PC} - and \mathbf{PC}_{fin} -spaces can be characterized by the same cancellation law. This is neither the case for separated (super) convex spaces ([1], 2.3, 3.6) nor for separated absolutely (totally) convex spaces ([8], 11.2, 13.8). \square

§4 No Cogenerator for Preseparated \mathbf{PC} -Spaces

By 3.6, the category \mathbf{PC}_{sep} has a cogenerator, namely $\hat{\Delta}(\mathbb{R})$. \mathbf{PC}_{psep} is the full subcategory of \mathbf{PC} spanned by all preseparated positively convex spaces, i.e. such spaces, where $\alpha x = \alpha y$ implies $x = y$ ($x, y \in D, 0 < \alpha < 1$) ([14], 4.1). $\overline{\mathbb{R}^+} := \mathbb{R}^+ \cup \{\infty\}$ (where $\infty \notin \mathbb{R}^+$) is a positively convex space with zero element 0 in the following way:

Let $\sum_i \alpha_i x_i := \infty$ if $x_{i_0} = \infty$ for some $i_0 \in supp\alpha$ or if $x_i \in \mathbb{R}^+$ for all $i_0 \in supp\alpha$ and (the real sum) $\sum_i \alpha_i x_i$ is divergent. If the series $\sum_i \alpha_i x_i$ converges in \mathbb{R}^+ take this as the value of the formal sum ($\alpha \in \Omega^+, x_i \in \overline{\mathbb{R}^+}$ ($i \in \mathbb{N}$))

([4], p. 2) and ([5], 1.11, [2], 1.2)).

For any set J , a **PC**-congruence relation “ \sim ” can be defined on the cartesian power $(\overline{\mathbb{R}^+})^J$ by $x \sim y$ if and only if $x = y$ or there exist $k, j \in J$ with $x_k = y_j = \infty$ ($x = (x_j)_{j \in J}$, $y = (y_j)_{j \in J} \in (\overline{\mathbb{R}^+})^J$) ([4], p. 2). Let $\pi : (\overline{\mathbb{R}^+})^J \longrightarrow (\overline{\mathbb{R}^+})^J / \sim$ be the canonical projection and $x, y \in (\overline{\mathbb{R}^+})^J$, $0 < \alpha < 1$ with $\pi(\alpha x) = \alpha \pi(x) = \alpha \pi(y) = \pi(\alpha y)$. This implies $\alpha x = \alpha y$ (and thus $x = y$) or the existence of elements $k, j \in J$ with $\alpha x_k = (\alpha x)_k = \infty = (\alpha y)_j = \alpha y_j$, and this leads to $x_k = \infty = y_j$. Hence $(\overline{\mathbb{R}^+})^J / \sim$ is preseparated.

For any (preseparated) positively convex space D , any infinite set J with $\text{card } D < \text{card } J$ and every **PC**-morphism $f : (\overline{\mathbb{R}^+})^J / \sim \longrightarrow D$, $f(\pi((1)_{j \in J})) = f(\pi((\infty)_{j \in J}))$ holds ([4], p. 2). This leads to

4.1 Corollary: The category \mathbf{PC}_{psep} does not have a cogenerator. □

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STRONG ORTHOGONALITY, ALMOST MEMBERSHIP, AND CONSERVATION OF INVARIANCE AND MULTIPLICATIVITY IN CONNECTION WITH STABLE SETS OF MEASURES

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Abstract

Let M denote some \leq -stable set of finite and finitely additive measures on some algebra \mathcal{A} of subsets of some set and let M_Σ stand for the set consisting of all finite and finitely additive measures on \mathcal{A} of the type $\sum_{j=1}^{\infty} \mu_j$, $\mu_j \in M$, $j = 1, 2, \dots$. Furthermore, $\mu \in_{\nu} M_\Sigma$ means that for all $\varepsilon > 0$ there exists some $A \in \mathcal{A}$ satisfying $\nu(A^c) \leq \varepsilon$ and $\mu_A \in M_\Sigma$, μ and ν being finite and finitely additive measures on \mathcal{A} . It is shown that $\mu \in_{\nu} M_\Sigma$ is valid for all two-valued, finite, and finitely additive measures ν on \mathcal{A} if and only if $\mu \in M$ holds true. Furthermore, the property of any finite and finitely additive measure μ on \mathcal{A} to satisfy $\mu_1(B) = \sup\{\mu_A(B) : \mu_A \in M, A \in \mathcal{A}\}$, $B \in \mathcal{A}$, for the part μ_1 of the decomposition of μ according to $\mu = \mu_1 + \mu_2$, $\mu_1 \in M$, $\mu_2 \in M^\perp$, where M is in addition \sum -stable, is characterized by strong orthogonality. Finally, it is shown that invariance of any finite and finitely additive measure μ on \mathcal{A} is hereditary for the component $\mu_1 \in M$ of μ and that direct products of finite and finitely additive measures have a property of multiplicativity concerning their components belonging to M .

1 Terminology and auxiliary results

Let $ba_+(\Omega, \mathcal{A})$ denote the set consisting of all finite (bounded) and (finitely) additive measures μ defined on \mathcal{A} , i.e. $\mu : \mathcal{A} \rightarrow \mathbb{R}$ satisfies $\mu(A) \geq 0$, $A \in \mathcal{A}$, and $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$, $A_j \in \mathcal{A}$, $j = 1, 2$, $A_1 \cap A_2 = \emptyset$. Furthermore, a subset M of $ba_+(\Omega, \mathcal{A})$ is called \leq -stable if and only if $\nu \leq \mu$ for $\mu \in M$ and $\nu \in ba_+(\Omega, \mathcal{A})$ implies $\nu \in M$. Finally, a subset M of $ba_+(\Omega, \mathcal{A})$ is said to be \sum -stable if and only if $M = M_\Sigma$ is valid, where M_Σ stands for the set of all elements of $ba_+(\Omega, \mathcal{A})$ of the type $\sum_{j=1}^{\infty} \mu_j$, $\mu_j \in M$, $j = 1, 2, \dots$. The starting point is the following Riesz type decomposition (cf. [3]), which will be proved here for sake of completion, where M^\perp is introduced by $\{\mu \in ba_+(\Omega, \mathcal{A}) : \mu \perp \nu, \nu \in M\}$ and where $\mu \perp \nu$ means that the greatest lower bound $\mu \wedge \nu$ of μ and ν with respect to the natural ordering (i.e. \leq setwise) of $ba_+(\Omega, \mathcal{A})$ is equal to 0, i.e. for $\varepsilon > 0$ there exists some $A \in \mathcal{A}$ satisfying $\mu(A) \leq \varepsilon$ and $\nu(A^c) \leq \varepsilon$, since $\mu \wedge \nu$ might be described by $(\mu \wedge \nu)(A) = \inf\{\mu(A_1) + \nu(A_2) : A_1 \cup A_2 = A, A_1 \cap A_2 = \emptyset, A_j \in \mathcal{A}, j = 1, 2\}$, $A \in \mathcal{A}$.

Theorem 1 (*Riesz type decomposition*)

Any $\mu \in ba_+(\Omega, \mathcal{A})$ can be uniquely decomposed according to $\mu = \mu_1 + \mu_2$, $\mu_1 \in M_\Sigma$, $\mu_2 \in M^\perp$, where M is some non-empty and \leq -stable subset of $ba_+(\Omega, \mathcal{A})$.

Proof: Let \mathcal{M} denote the set consisting of all subsets $\{\nu_k : k \in K\}$ of $ba_+(\Omega, \mathcal{A})$ satisfying $\sum_{k \in K} \nu_k \leq \mu$, where $\sum_{k \in K} \nu_k$ stands for the least upper bound of all finite sums $\sum_{i \in F} \nu_i$ with respect to the natural ordering of $ba_+(\Omega, \mathcal{A})$, where F is some finite subset of K . Then there exists some maximal element $\{\nu_i : i \in I\}$ of \mathcal{M} and μ_1 might be introduced by $\sum_{i \in I} \nu_i$, where $\sum_{i \in I} \nu_i$ might be written as an ordinary sum, if one takes IV.11.7 of [2] concerning the order completeness of L_1 together with some Stonian representation argument according to IV.9.11 of [2] into consideration. Introducing μ_2 by $\mu - \mu_1$ results in the existence of the Riesz type decomposition, whereas the uniqueness assertion concerning μ_1 and μ_2 might be shown by applying a Jordan decomposition argument as follows: $\mu = \mu_1 + \mu_2 = \mu'_1 + \mu'_2$, $\mu_1, \mu'_1 \in M_\Sigma$, $\mu_2, \mu'_2 \in M^\perp$ implies the following equation for the positive parts of $\mu_1 - \mu'_1$ and $\mu'_2 - \mu_2$ of the corresponding Jordan decomposition: $(\mu_1 - \mu'_1)^+ = (\mu'_2 - \mu_2)^+$, from which $(\mu_1 - \mu'_1)^+ \in M^\perp$ because of $(\mu_1 - \mu'_1)^+ \leq \mu'_2 \in M^\perp$ follows. Finally, $(\mu_1 - \mu'_1)^+ \leq \mu_1$ together with $\mu_1 \perp \lambda$ for all $\lambda \in M^\perp$ results in $(\mu_1 - \mu'_1)^+ = 0$, from which $(\mu'_1 - \mu_1)^+ = (\mu_1 - \mu'_1)^- = 0$ follows, where $(\mu_1 - \mu'_1)^-$ stands for the negative part of $\mu_1 - \mu'_1$, i.e. $\mu_1 = \mu'_1$ and $\mu_2 = \mu'_2$ holds true. \square

Remark: As an application of this Riesz type decomposition one might rederive the Hammer-Sobczyk decomposition (cf. [1], p. 146) by introducing M as the set consisting of all two-valued elements belonging to $ba_+(\Omega, \mathcal{A})$. The characterization of the elements of M^\perp as the strongly continuous finitely additive, and finite measures on \mathcal{A} might be derived by a Stonian representation argument (cf. [2], IV.9.11 and [4], p. 373).

2 Strong Orthogonality

The theorem about a version of the Riesz decomposition theorem admits the following decomposition result, where μ_A for $\mu \in ba_+(\Omega, \mathcal{A})$ and $A \in \mathcal{A}$ is defined by $\mu_A(B) = \mu(A \cap B)$, $B \in \mathcal{A}$, and $\mu \in ba_+(\Omega, \mathcal{A})$ is said to be strongly orthogonal with respect to $\nu \in ba_+(\Omega, \mathcal{A})$ if and only if for any $\varepsilon > 0$ there exists some $A \in \mathcal{A}$ such that $\mu(A_\varepsilon^c) \leq \varepsilon$ and $\nu(A_\varepsilon) = 0$ is fulfilled. Furthermore, $M \oplus M^\perp$ is introduced as $\{\mu_1 + \mu_2 : \mu_1 \in M, \mu_2 \in M^\perp\}$ for any \leq -stable and Σ -stable subset M of $ba_+(\Omega, \mathcal{A})$.

Theorem 2 (*Strong orthogonality*)

Let M stand for some non-empty, \leq -stable and Σ -stable subst of $ba_+(\Omega, \mathcal{A})$. Then the following assertions hold true:

- (i) Any $\mu \in ba_+(\Omega, \mathcal{A})$ can be written uniquely as $\mu = \mu_1 + \mu_2$, $\mu_1 \in M$, $\mu_2 \in M^\perp$, i.e. $ba_+(\Omega, \mathcal{A}) = M \oplus M^\perp$, $M \cap M^\perp = \{0\}$.
- (ii) $M^{\perp\perp} = M$.
- (iii) μ_1 occurring in (i) has the representation $\mu_1(B) = \sup\{\mu_A(B) : \mu_A \in M, A \in \mathcal{A}\}$, $B \in \mathcal{A}$, if and only if μ_1 is strongly orthogonal with respect to μ_2 occurring in (i).

Proof: Part (i) is an immediate consequence of the Riesz type decomposition theorem, whereas (ii) is an easy application of part (i). For the proof of (iii) one derives from the equation $\mu_1(B) = \sup\{\mu_A(B) : \mu_A \in M, A \in \mathcal{A}\}$, $B \in \mathcal{A}$, that for any $\varepsilon > 0$ there exists some $A_\varepsilon \in \mathcal{A}$ satisfying $\mu_1(\Omega) \leq \mu_{A_\varepsilon}(\Omega) + \varepsilon$ and $\mu_{A_\varepsilon} \in M$, from which $\mu_{A_\varepsilon} = \mu_{1,A_\varepsilon}$ and $\mu_2(A_\varepsilon) = 0$ follows because of $\mu_2 \perp \mu_{A_\varepsilon}$. Therefore, one arrives at $\mu_1(A_\varepsilon^c) = \mu_1(\Omega) - \mu_1(A_\varepsilon) = \mu_1(\Omega) - \mu_{A_\varepsilon}(\Omega) \leq \varepsilon$, i.e. μ_1 is strongly orthogonal with respect to μ_2 . Conversely, strong orthogonality of μ_1 with respect to μ_2 yields $\mu_1(A_\varepsilon \cap B) = \mu(A_\varepsilon \cap B)$, $B \in \mathcal{A}$, where $A_\varepsilon \in \mathcal{A}$ fulfills $\mu_2(A_\varepsilon) = 0$ and $\mu_1(A_\varepsilon^c) \leq \varepsilon$. Therefore, $\mu_1(B) = \mu_1(B \cap A_\varepsilon) + \mu_1(B \cap A_\varepsilon^c) \leq \mu_{A_\varepsilon}(B) + \varepsilon$, $B \in \mathcal{A}$, and $\mu_{A_\varepsilon} \in M$ is valid, from which $\mu_1(B) = \sup\{\mu_A(B) : \mu_A \in M, A \in \mathcal{A}\}$, $B \in \mathcal{A}$, follows. \square

Remark: The special case $M = M_\sigma$, where M_σ consists of the σ -additive (countably additive) elements of $ba_+(\Omega, \mathcal{A})$ one gets the Hewitt-Yosida decomposition $ba_+(\Omega, \mathcal{A}) = M_\sigma \oplus M_\sigma^\perp$. The elements of M_σ^\perp are called purely finitely additive. In particular, strong orthogonality in this case of any element of M_σ with respect to any element of M_σ^\perp if \mathcal{A} is in addition a σ -algebra of subsets of Ω (cf. [1], p. 244), shows that part (iii) is valid for the decomposition of Hewitt-Yosida.

The preceding remark together with the theorem about strong orthogonality admits the following

Corollary 1 (Strong orthogonality)

Let M be a non-void, \leq -stable, and Σ -stable subset of $ba_+(\Omega, \mathcal{A})$ and let \tilde{M} stand for the set consisting of all elements of M being σ -additive, where \mathcal{A} is a σ -algebra of subsets of Ω . Then \tilde{M} is \leq -stable and Σ -stable and any $\mu \in ba_+(\Omega, \mathcal{A})$ can be uniquely decomposed according to $\mu = \mu_1 + \mu_2$, $\mu_1 \in \tilde{M}$, $\mu_2 \in \tilde{M}^\perp$, and μ_1 can be represented as $\mu_1(B) = \sup\{\mu_A(B) : \mu_A \in \tilde{M}, A \in \mathcal{A}\}$, $B \in \mathcal{A}$.

Proof: Clearly \tilde{M} is \leq -stable, Σ -stable and non-empty because of $0 \in \tilde{M}$. According to the theorem about strong orthogonality it remains to show that $\mu_1 \perp \mu_2$ implies that μ_1 is strongly orthogonal relative to μ_2 , which might be seen as follows: For any $\varepsilon > 0$ there exist sets $A_{n,\varepsilon} \in \mathcal{A}$ satisfying $\mu_1(A_{n,\varepsilon}) \leq \frac{\varepsilon}{2^n}$ and $\mu_2(A_{n,\varepsilon}^c) \leq \frac{\varepsilon}{2^n}$, $n \in \mathbb{N}$, from which $\mu_1(\bigcup_{n=1}^{\infty} A_{n,\varepsilon}) \leq \varepsilon$ and $\mu_2((\bigcup_{n=1}^{\infty} A_{n,\varepsilon})^c) = 0$ follows, since μ_1 is sub- σ -additive and \mathcal{A} is a σ -algebra. \square

Remark: A further application of the decomposition result in connection with strong orthogonality concerns the subset M_λ of $ba_+(\Omega, \mathcal{A})$ consisting of all $\mu \in ba_+(\Omega, \mathcal{A})$ being absolutely continuous with respect to $\lambda \in ba_+(\Omega, \mathcal{A})$, i.e. for any $\varepsilon > 0$ there exists some $\delta > 0$ such that $\lambda(A) < \delta$ for some $A \in \mathcal{A}$ implies $\mu(A) < \varepsilon$ (notation: $\lambda \ll \mu$). Clearly, M is \leq -stable and Σ -stable, whereas M^\perp consists of all $\nu \in ba_+(\Omega, \mathcal{A})$ satisfying $\nu \perp \lambda$, since $\lambda \in M$ and any $\nu \in ba_+(\Omega, \mathcal{A})$ with the property $\nu \perp \lambda$ fulfills $\nu \in M^\perp$. This last assertion follows from the observation that $\nu \wedge \lambda$ is absolutely continuous with respect to $\nu \wedge \lambda$ for any $\mu \in M_\lambda$, which might be seen by the explicit representation of $\mu_1 \wedge \mu_2$, $\mu_j \in ba_+(\Omega, \mathcal{A})$, $j = 1, 2$, which has been introduced in the first section. Thus one arrives at a Lebesgue type decomposition for finite and finitely additive measures, i.e. $ba_+(\Omega, \mathcal{A}) = M_\lambda \oplus M_\lambda^\perp$. Furthermore, any \leq -stable and Σ -stable subset M of $ba_+(\Omega, \mathcal{A})$ is also \ll -stable, i.e. $\mu \ll \lambda$ for $\mu \in ba_+(\Omega, \mathcal{A})$ and $\lambda \in M$ implies $\mu \in M$. This follows

from the decomposition $\mu = \mu_1 + \mu_2$, $\mu_1 \in M$, $\mu_2 \in M^\perp$, since $\mu_2 \ll \lambda$ and $\mu_2 \perp \lambda$ yields $\mu_2(\Omega) = 0$.

3 Almost Membership and Conservation of Invariance

In this section the starting point is the notion $\mu \in_{\nu} M_\Sigma$ for $\mu, \nu \in ba_+(\Omega, \mathcal{A})$ and M being some non-empty and \leq -stable subset of $ba_+(\Omega, \mathcal{A})$, which is defined by the existence of some $A_\varepsilon \in \mathcal{A}$ for any $\varepsilon > 0$ satisfying $\mu_{A_\varepsilon} \in M_\Sigma$ and $\nu(A_\varepsilon^c) \leq \varepsilon$. The corresponding characterization of $\mu \in M_\Sigma$ by $\mu \in_{\nu} M_\Sigma$ for sufficient many $\nu \in ba_+(\Omega, \mathcal{A})$ is summarized by the following

Theorem 3 (*Almost membership*)

Let M stand for some non-void and \leq -stable subset of $ba_+(\Omega, \mathcal{A})$. Then $\mu \in M_\Sigma$ is valid if and only if $\mu_{\nu} \in M_\Sigma$ holds true for all two-valued $\nu \in ba_+(\Omega, \mathcal{A})$.

Proof: Clearly, $\mu \in M_\Sigma$ implies $\mu \in_{\nu} M_\Sigma$ for all $\nu \in M_\Sigma$ because of $\mu_A \in M_\Sigma$, $A \in \mathcal{A}$, since $\mu = \sum_{i=1}^{\infty} \mu_j$, $\mu_j \in M$, $j = 1, 2, \dots$, results by means of \leq -stability of M in $\mu_A = \sum_{i=1}^{\infty} \mu_{j,A}$, $\mu_{j,A} \in M$, $A \in \mathcal{A}$, $j = 1, 2, \dots$. For the proof of the converse direction one might start from the observation that for any $\mu \in ba_+(\Omega, \mathcal{A})$ with $\mu(\Omega) > 0$ there exists some two-valued $\nu \in ba_+(\Omega, \mathcal{A})$, $\nu(\Omega) > 0$, such that $\mu(A) = 0$ for some $A \in \mathcal{A}$ implies $\nu(A) = 0$. This might be seen by introducing the sub-algebra \mathcal{A}' of \mathcal{A} defined by $\{A \in \mathcal{A} : \mu(A) = 0 \text{ or } \mu(A) = \mu(\Omega)\}$. Any $\{0, \mu(\Omega)\}$ -valued extension $\nu \in ba_+(\Omega, \mathcal{A})$ (cf. [4], p. 372) of $\mu|_{\mathcal{A}'}$ has the corresponding property. Let now ν stand for some two-valued element of $ba_+(\Omega, \mathcal{A})$ with $\nu(\Omega) > 0$ such that $\mu_2(A) = 0$ for some $A \in \mathcal{A}$ results in $\nu(A) = 0$, where $\mu \in ba_+(\Omega, \mathcal{A})$ satisfies $\mu \in_{\nu} M_\Sigma$, $\mu = \mu_1 + \mu_2$, $\mu_1 \in M_\Sigma$, $\mu_2 \in M^\perp$, is valid. Then $\mu_{A_\varepsilon} \in M_\Sigma$ for some $A_\varepsilon \in \mathcal{A}$ satisfying $\nu(A_\varepsilon^c) \leq \varepsilon$ for any given $\varepsilon > 0$ and $\varepsilon < \nu(\Omega)$, leads to $\nu(A_\varepsilon^c) = 0$ and $\mu_{2,A_\varepsilon} = 0$ because of $\mu_{2,A} \in M^\perp$ for any $A \in \mathcal{A}$. Therefore, $\nu(A_\varepsilon) = 0$ together with $\nu(A_\varepsilon^c) = 0$ produces the contradiction $\nu(\Omega) = 0$. \square

A further topic of this section concerns the question of conservation of invariance under some family \mathcal{F} of $(\mathcal{A}, \mathcal{A})$ -measurable functions $f : \Omega \rightarrow \Omega$, i.e. $f^{-1}(\mathcal{A}) \subset \mathcal{A}$ is satisfied. Any $f \in \mathcal{F}$ and $\mu \in ba_+(\Omega, \mathcal{A})$ induces some $\mu^f \in ba_+(\Omega, \mathcal{A})$ defined by $\mu^f(A) = \mu(f^{-1}(A))$, $A \in \mathcal{A}$.

Theorem 4 (*Conservation of invariance*)

Let M denote some non-empty and \leq -stable subset of $ba_+(\Omega, \mathcal{A})$ and \mathcal{F} some non-empty set of functions $f : \Omega \rightarrow \Omega$ satisfying $f^{-1}(\mathcal{A}) \subset \mathcal{A}$, such that $\nu^f \in M$ is satisfied for any $\nu \in M$ and $f \in \mathcal{F}$. Then $\mu = \mu^f$, $f \in \mathcal{F}$, for some $\mu \in ba_+(\Omega, \mathcal{A})$ implies $\mu_j = \mu_j^f$, $f \in \mathcal{F}$, $j = 1, 2$, where $\mu = \mu_1 + \mu_2$, $\mu_1 \in M_\Sigma$, $\mu_2 \in M^\perp$, holds true.

Proof: The representation $\mu_1 = \sum_{j=1}^{\infty} \nu_j$, $\nu_j \in M$, $j = 1, 2, \dots$, implies $\mu_1^f = \sum_{j=1}^{\infty} \nu_j^f$, $f \in \mathcal{F}$, and, therefore, one arrives together with $\nu_j^f \in M$, $f \in \mathcal{F}$, $j = 1, 2, \dots$, and $\mu = \mu^f$, $f \in \mathcal{F}$, at $\sum_{j=1}^{\infty} \nu_j \geq \sum_{j=1}^{\infty} \nu_j^f$, if one takes into consideration the proof of the existence of the Riesz type decomposition introduced in the first section, i.e. $\mu_1(A) = \sup\{\sum_{j=1}^n \nu_j(A) : \nu_j \in M, j = 1, \dots, n, \sum_{j=1}^n \nu_j \leq \mu, n \in \mathbb{N}\}$, $A \in \mathcal{A}$. Now

$\sum_{j=1}^{\infty} \nu_j(\Omega) = \sum_{j=1}^{\infty} \nu_j^f(\Omega)$, $f \in \mathcal{F}$, yields $\sum_{j=1}^{\infty} \nu_j = \sum_{j=1}^{\infty} \nu_j^f$, $f \in \mathcal{F}$, i.e. $\mu_1 = \mu_1^f$, $f \in \mathcal{F}$, and, therefore, also $\mu_2 = \mu_2^f$, $f \in \mathcal{F}$. \square

4 Conservation of Multiplicativity

Let \mathcal{A}_j denote some algebra of subsets of some set Ω_j and M_j some subsets of $ba_+(\Omega_j, \mathcal{A}_j)$, $j = 1, 2$, being \leq -stable and Σ -stable. Then M_{12} consisting of all $\mu \in ba_+(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$, such that the marginal measures $\mu^{(j)}$, $j = 1, 2$, of μ belong to M_j , $j = 1, 2$, is \leq -stable and Σ -stable. Here, $\mathcal{A}_1 \otimes \mathcal{A}_2$ stands for the algebra of subsets of $\Omega_1 \times \Omega_2$ generated by $\{\mathcal{A}_1 \times \mathcal{A}_2 : A_j \in \mathcal{A}_j, j = 1, 2\}$, i.e. $\mathcal{A}_1 \otimes \mathcal{A}_2 = \{\sum_{j=1}^n (A_{1j} \times A_{2j}) : A_{ij} \in \mathcal{A}_i, j = 1, \dots, n, i = 1, 2, n \in \mathbb{N}\}$ and $\mu^{(1)}, \mu^{(2)}$ are defined by $\mu^{(1)}(A_1) = \mu(A_1 \times \Omega_2)$, $A_1 \in \mathcal{A}_1$, $\mu^{(2)}(A_2) = \mu(\Omega_1 \times A_2)$, $A_2 \in \mathcal{A}_2$. Furthermore, $\mu_1 \otimes \mu_2 \in ba_+(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ is defined for $\mu_j \in ba_+(\Omega_j, \mathcal{A}_j)$, $j = 1, 2$, by $(\mu_1 \otimes \mu_2)(\sum_{j=1}^n (A_{1j} \times A_{2j})) = \sum_{j=1}^n \mu_1(A_{1j})\mu_2(A_{2j})$, $A_{ij} \in \mathcal{A}_i$, $j = 1, \dots, n$, $i = 1, 2$, $n \in \mathbb{N}$. Now everything is prepared for the following

Theorem 5 (Conservation of multiplicativity)

Let \mathcal{A}_j stand for some algebra of subsets of a set Ω_j and let M_j denote subsets of $ba_+(\Omega_j, \mathcal{A}_j)$ being \leq -stable and Σ -stable, $j = 1, 2$. Then the following assertions hold true:

- (i) The component of $\mu_1 \otimes \mu_2$ for any $\mu_j \in ba_+(\Omega_j, \mathcal{A}_j)$, $j = 1, 2$, of the decomposition of $\mu_1 \otimes \mu_2$ belonging to M according to $ba_+(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) = M \otimes M^\perp$, where M is some \leq -stable and Σ -stable subset of $ba_+(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$, containing every $\mu_1 \otimes \mu_2$, $\mu_j \in M_j$, $j = 1, 2$, is equal to $\mu_{11} \otimes \mu_{21}$. Here $\mu_j = \mu_{j1} + \mu_{j2}$, $\mu_{j1} \in M_j$, $\mu_{j2} \in M_j^\perp$, $j = 1, 2$.
- (ii) $M_1 \otimes M_2$ introduced as the intersection of all \leq -stable and Σ -stable subsets of $ba_+(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ containing any $\mu_1 \otimes \mu_2$, $\mu_j \in M_j$, $j = 1, 2$, is \leq -stable and Σ -stable and is properly contained in M_{12} in the case where $\Omega_1 = \Omega_2 = \Omega$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, \mathcal{A} being some σ -algebra of subsets of Ω containing all singletons $\{\omega\}$, $\omega \in \Omega$, and where $M_1 = M_2 = M$, M being some \leq -stable and Σ -stable subset of $ba_+(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ containing some $\mu_0 \in ba_+(\Omega, \mathcal{A})$ being σ -additive and fulfilling $\mu_0(\{\omega\}) = 0$, $\omega \in \Omega$, $\mu_0(\Omega) = 1$.

Proof: $M_1 \otimes M_2$ is well-defined since M_{12} introduced at the beginning of this section belongs to the corresponding intersection defining $M_1 \otimes M_2$. Furthermore, $M_1 \otimes M_2$ is \leq -stable and Σ -stable and contains every $\mu_1 \otimes \mu_2$, $\mu_j \in M_j$, $j = 1, 2$. Thus, there are at least two subsets M of $ba_+(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$, which are \leq -stable, Σ -stable, and contain all $\mu_1 \otimes \mu_2$, $\mu_j \in M_j$, $j = 1, 2$. In particular, $\mu_{11} \otimes \mu_{21} \in M$ is valid and part (i) of the theorem follows from the observation $\lambda_1 \otimes \lambda_2 \in M^\perp$ for $\lambda_j \in ba_+(\Omega_j, \mathcal{A}_j)$, $j = 1, 2$, satisfying $\lambda_1 \in M_1^\perp$, since M^\perp is Σ -stable. The assertion $\lambda_1 \otimes \lambda_2 \in M^\perp$ might be derived from $\lambda_1 \perp \mu^{(1)}$ or $\lambda_2 \perp \mu^{(2)}$, where the $\mu^{(j)}$, $j = 1, 2$, are associated with some $\mu \in M$, since $\lambda_1 \otimes \lambda_2 \perp \mu$ holds true.

For the proof of (ii) let $\mu \in ba_+(\Omega \times \Omega, \mathcal{A} \otimes \mathcal{A})$ be (well-) defined by $\mu(\sum_{j=1}^n (A_{1j} \times A_{2j})) = \sum_{j=1}^n \mu_0(A_{1j} \cap A_{2j})$, $A_{1j}, A_{2j} \in \mathcal{A}$, $j = 1, \dots, n$, $n \in \mathbb{N}$, where $\mu_0 \in ba_+(\Omega, \mathcal{A})$ is σ -additive and satisfies $\mu_0(\{\omega\}) = 0$, $\omega \in \Omega$, $\mu_0(\Omega) = 1$. Therefore, μ defined on $\mathcal{A} \otimes \mathcal{A}$

is also σ -additive and fulfills $\mu(\{(\omega, \omega)\}) = 0$, $\omega \in \Omega$, $\mu(\Omega \times \Omega) = 1$. Finally μ might be extended uniquely to the σ -algebra $\sigma(\mathcal{A} \otimes \mathcal{A})$ generated by $\mathcal{A} \otimes \mathcal{A}$ as a probability measure ν according to $\nu(C) = \inf\{\sum_{j=1}^{\infty} \mu(A_{1j} \times A_{2j}) : C \subset \sum_{j=1}^{\infty} (A_{1j} \times A_{2j}), A_{1j}, A_{2j} \in \mathcal{A}, j = 1, 2, \dots\}$, $C \in \sigma(\mathcal{A} \otimes \mathcal{A})$, from which $\nu^*(\Delta^c) = 0$, i.e. $\nu_*(\Delta) = 1$ follows, where Δ stands for the diagonal $\{(\omega, \omega) : \omega \in \Omega\}$ of $\Omega \times \Omega$, and ν^* resp. ν_* denotes the outer resp. inner probability measure of ν . Let now $M^{(\mu)}$ stand for the set consisting of all $\lambda \in ba_+(\Omega \times \Omega, \mathcal{A} \otimes \mathcal{A})$ satisfying $\lambda \perp \mu$. Then $M^{(\mu)}$ is \leq -stable, Σ -stable, and contains every $\mu_1 \otimes \mu_2$, $\mu_j \in M$, $j = 1, 2$. This last assertion $\mu_1 \otimes \mu_2 \in M^{(\mu)}$, $\mu_j \in M$, $j = 1, 2$, might be seen by introducing the Hewitt-Yosida decomposition $\mu_1 = \mu_{1\sigma} + \mu_{1p}$, $\mu_2 = \mu_{2\sigma} + \mu_{2p}$ of μ_j , $j = 1, 2$, i.e. $\mu_{j\sigma} \in ba_+(\Omega, \mathcal{A})$, $j = 1, 2$, is σ -additive and $\mu_{jp} \in ba_+(\Omega, \mathcal{A})$ is purely finitely additive, i.e. $\mu_{jp} \perp \lambda$, $j = 1, 2$, holds true for any $\lambda \in ba_+(\Omega, \mathcal{A})$ being σ -additive. Now $\nu_*(\Delta) = 1$ and $\nu(\{(\omega, \omega)\}) = 0$, $\omega \in \Omega$, implies $\mu \perp \mu_{1\sigma} \otimes \mu_{2\sigma}$ and $\mu_0 \perp \mu_{jp}$, $j = 1, 2$, results in $\mu \perp \mu_{1p} \otimes \mu_{2p}$, $\mu \perp \mu_{1p} \otimes \mu_{2\sigma}$, and $\mu \perp \mu_{1\sigma} \otimes \mu_{2p}$, i.e. $\mu \perp \mu_1 \otimes \mu_2$. However, $\mu \notin M^{(\mu)}$ because of $\mu(\Omega) = 1$, i.e. $\mu \notin M \otimes M$ is valid, whereas $\mu \in M_{12}$ holds true. \square

Remark: Part (ii) of the theorem about multiplicativity shows that the inclusion $M_1^\perp \otimes M_2^\perp \subset (M_1 \otimes M_2)^\perp$ is in general proper, since otherwise $M_1^\perp \otimes M_2^\perp \subset M_{12}^\perp \subset (M_1 \otimes M_2)^\perp$ results in $M_{12} = M_{12}^{\perp\perp} \subset (M_1^\perp \otimes M_2^\perp)^\perp = M_1^{\perp\perp} \otimes M_2^{\perp\perp} = M_1 \otimes M_2$.

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On the complete r -partite graphs and their line graphs

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Abstract

Motivated by the study of degeneracy graphs we investigate the properties of complete r -partite graphs and their line graphs.

Keywords: Degeneracy graphs, multipartite graphs, line graphs, connectivities of a graph.

0. Introduction and basic concepts

In order to solve degeneracy problems in linearly constrained optimization the so-called degeneracy graphs, assigned to a degenerate vertex x of the feasible solution set, have proved to be useful [4,5,7,10,11]. In the special case when the degeneracy degree of x is 2 (i.e. two basic variables equal zero), degeneracy graphs are the line graphs of complete multipartite graphs.

To date only the line graphs of special complete multipartite graphs have been investigated [2,p.286, 6,8]. In this paper we will study the properties of general complete r -partite graphs and their line graphs. While papers dealing with connectivities of graphs usually develop lower or upper bounds [1, 3, 9], we in particular derive formulae for the connectivities of $K(p_1, \dots, p_r)$ and $L(K(p_1, \dots, p_r))$.

The *complete r-partite graph* $K(p_1, \dots, p_r)$ is that graph with node set partition

$$V = V_1 \cup \dots \cup V_r$$

$$(r \geq 2, |V_i| = p_i \text{ for } i = 1, \dots, r),$$

where two nodes are connected by an edge iff they lie in different components of V (cf. Fig.1).

Let G denote a finite undirected graph without loops or multiple edges. The *line graph* $L(G)$ of G is that graph whose nodes correspond to the edges of G such that two nodes of $L(G)$ are adjacent iff the corresponding edges of G are adjacent (cf. Fig. 2).

The connectivities can be - equivalent to the usual way - defined as follows (cf. e.g. [3, p. 170]):

A graph G is said to be *n-connected* (*m-edge connected*) iff between each pair of distinct nodes there exist at least n disjoint (m edge-disjoint) paths.

The *connectivity* κ (the *edge connectivity* λ) of G is the maximum integer n (or m) for which G is *n-connected* (*m-edge connected*). An immediate consequence of this definition is

$$\kappa \leq \lambda \leq \delta$$

where δ denotes the *minimum degree* of the respective graph (cf. [3, p. 171]).

In this article a sequence of pairwise disjoint edges (e_1, \dots, e_n) , where e_i and e_{i+1} are adjacent for $i = 1, \dots, n-1$, will be called an *edge path*.

As usual, the *distance* $d(v, w)$ between two nodes v, w of a graph G is defined as the length of a shortest path from v to w . The longest distance between two nodes in G is called the *diameter* of G .

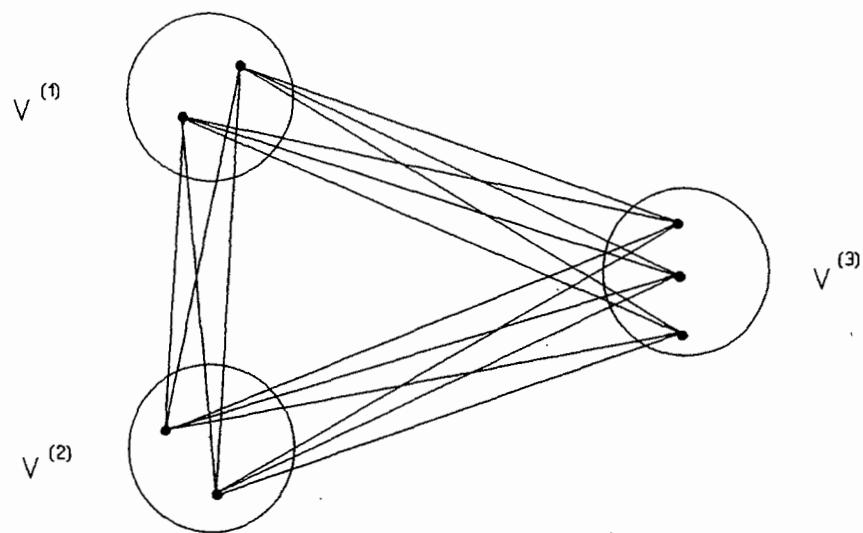


Fig.1: The complete r-partite graph $K(2,2,3)$

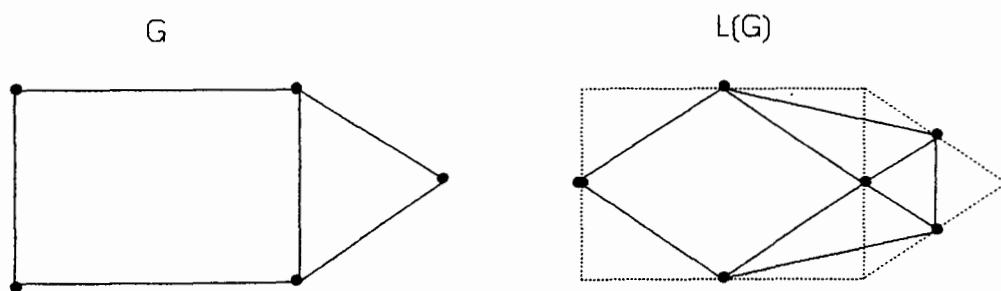


Fig. 2: A graph G and its line graph $L(G)$

1. Properties of $K(p_1, \dots, p_r)$

The number of edges of $K(p_1, \dots, p_r)$ is

$$\begin{aligned} & \sum_{1 \leq i < j \leq r} p_i p_j \\ &= \frac{1}{2} (n^2 - \sum_{i=1}^r p_i^2) , \end{aligned}$$

where $n = \sum_{i=1}^r p_i$, and the diameter of a complete r -partite graph is ≤ 2 .

Lemma 1.1: Let V_i, V_j denote two components of the node set partition of $K(p_1, \dots, p_r)$ with $r \geq 2$, $i \neq j$, $p_i = |V_i| \leq p_j = |V_j|$, $n = \sum p_i$.

Then there exist $n - p_j$ node disjoint paths between any two nodes v, w with $v \in V_i$ and $w \in V_j$.

Proof: Let v_1, \dots, v_{p_i} , w_1, \dots, w_{p_j} and $u_1, \dots, u_{n-p_i-p_j}$ denote the nodes of V_i, V_j and $V \setminus \{V_i \cup V_j\}$, respectively.

Then there exist the following $n - p_j$ node disjoint paths between v_1 and w_1 (cf. Fig. 3):

$$\begin{aligned} P &:= (v_1, w_1), \\ P_t &:= (v_1, u_t, w_1) \quad \text{for } t = 1, \dots, n-p_i-p_j, \\ P_t^* &:= (v_1, w_t, v_t, w_1) \quad \text{for } t = 2, \dots, p_i. \blacksquare \end{aligned}$$

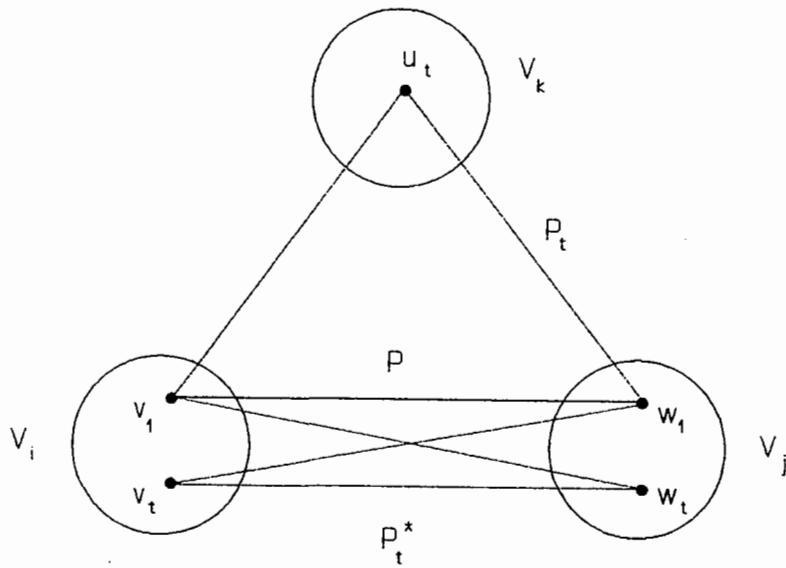


Fig. 3: Disjoint paths in the proof of Lemma 1.1

Proposition 1.2:

Let $G = K(p_1, \dots, p_r)$ be a complete r -partite graph with $r \geq 2$,

$p_1 \leq \dots \leq p_r$, $n = \sum p_i$. Then the connectivities κ , λ of G satisfy

$$\kappa = \lambda = \delta = n - p_r,$$

where δ denotes the minimum degree of G .

The proof follows immediately from Lemma 1.1.

2. Properties of $L(K(p_1, \dots, p_r))$

Let $e = \{v, w\}$ be an edge of $K(p_1, \dots, p_r)$ with $v \in V_i$ and $w \in V_j$, $p_i = |V_i|$, $p_j = |V_j|$, $i \neq j$. It is easy to prove that e - considered as a node of $L(K(p_1, \dots, p_r))$ - has degree $2n-2-p_i-p_j$. Let $g(v)$ denote the degree of the node v of $K(p_1, \dots, p_r)$. Since edges of a line graph $L(G)$ correspond to pairs of adjacent edges of G which can be classified according to the common node, the

number of edges of $L(K(p_1, \dots, p_r))$ is

$$\begin{aligned} & \sum_v \binom{g(v)}{2} \\ &= \sum_{i=1}^r \sum_{v \in V_i} \binom{g(v)}{2} \\ &= \sum_{i=1}^r p_i \binom{n-p_i}{2}. \end{aligned}$$

The diameter of $L(K(p_1, \dots, p_r))$ is ≤ 2 .

Lemma 2.1: Let $K(p_1, \dots, p_r)$ be a complete r -partite graph with $r \geq 2$ and $n = p_1 + \dots + p_r$. Moreover let $e = \{v_1, v_2\}$ and $e' = \{v_3, v_4\}$ denote two distinct edges of $K(p_1, \dots, p_r)$ such that

$$k^{(3)} + k^{(4)} \geq k^{(1)} + k^{(2)},$$

where $V^{(i)}$ is the component of the node set partition containing v_i , and $k^{(i)} := |V^{(i)}|$ denotes the number of nodes in $V^{(i)}$ ($i = 1, \dots, 4$).

Then there exist

$$2n - 2 - k^{(3)} - k^{(4)}$$

pairwise disjoint edge paths between e and e' .

Remark 2.2:

It seems reasonable to prove the assertion by induction on the number n of nodes. However, in the case

$$k^{(3)} + k^{(4)} = k^{(1)} + k^{(2)}$$

problems arise in the transition from $n-1$ to n .

Proof of Lemma 2.1: We will construct the above number of disjoint edge paths for the following cases:

Case 1:

$v^{(1)}, \dots, v^{(4)}$ are distinct components of V .

We obtain 4 disjoint edge paths of the form (cf. Fig. 4):

$$P_{v_i, v_j} := (\{v_1, v_2\}, \{v_i, v_j\}, \{v_3, v_4\})$$

for $i \in \{1, 2\}$, $j \in \{3, 4\}$.

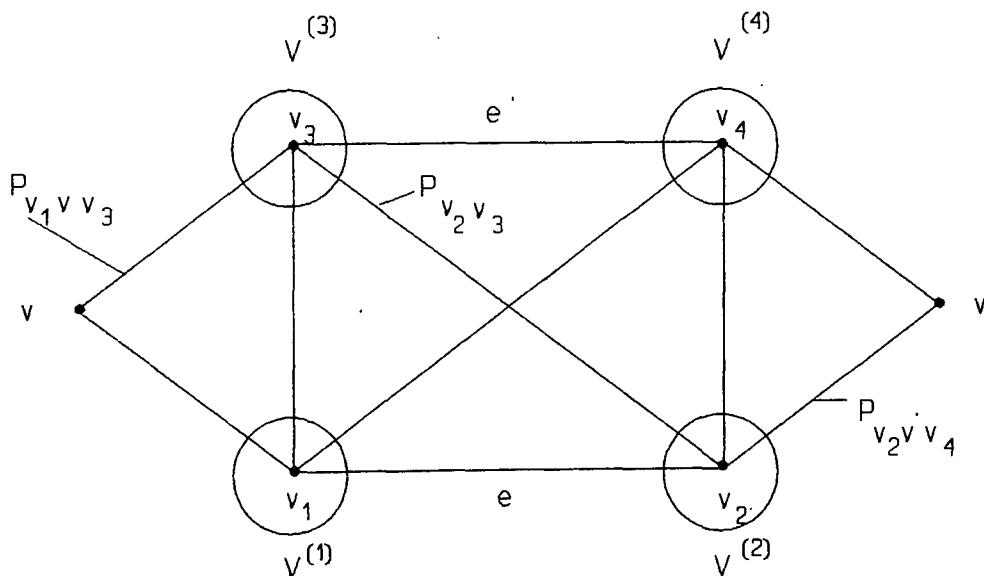


Fig. 4: Edge paths in case 1

Moreover, there are $n - k^{(1)} - k^{(3)} - 2$ edge paths of the form

$$P_{v_1, v_3} := (\{v_1, v_2\}, \{v_1, v\}, \{v_3, v\}, \{v_3, v_4\})$$

for $v \in V \setminus (V^{(1)} \cup V^{(3)} \cup \{v_2, v_4\})$

and $n - k^{(2)} - k^{(4)} - 2$ edge paths of the form

$$P_{v_2, v^*, v_4} := (\{v_1, v_2\}, \{v_2, v^*\}, \{v_4, v^*\} \{v_3, v_4\})$$

$$\text{for } v^* \in V \setminus (V^{(2)} \cup V^{(4)} \cup \{v_1, v_3\}) .$$

Finally we obtain $k^{(1)} + k^{(2)} - 2$ edge paths of the form (cf.

Fig. 5)

$$P^{(t)} := (\{v_1, v_2\}, \{v_{f(t)}, w_t'\}, \{w_t, w_t'\} \{w_t, v_{g(t)}\} \{v_3, v_4\})$$

$$\text{for } t = 1, \dots, k^{(1)} + k^{(2)} - 2 ,$$

where $w_1, \dots, w_{k^{(1)}+k^{(2)}-2}$ and $w'_1, \dots, w'_{k^{(3)}+k^{(4)}-2}$ denote the elements of $(V^{(1)} \cup V^{(2)}) \setminus \{v_1, v_2\}$ and $(V^{(3)} \cup V^{(4)}) \setminus \{v_3, v_4\}$, respectively, and

$$f(t) := \begin{cases} 1 & \text{if } w'_t \in V^{(3)} \\ 2 & \text{if } w'_t \in V^{(4)} , \end{cases}$$

$$g(t) := \begin{cases} 3 & \text{if } w_t \in V^{(1)} \\ 4 & \text{if } w_t \in V^{(2)} . \end{cases}$$

The above definitions of f and g insure that all edge paths are disjoint.

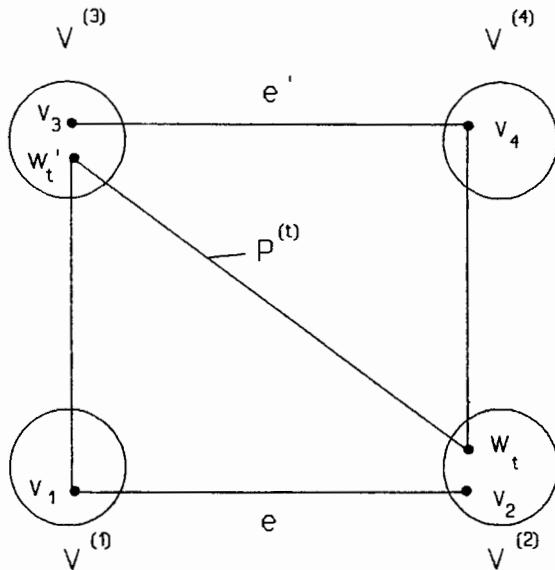


Fig. 5: Edge paths of the form $P^{(t)}$ in case 1

Case 2:

$$V^{(1)} = V^{(3)} ; \quad V^{(2)} \neq V^{(4)}$$

a) $v_1 \neq v_3$

We obtain the edge paths (cf. Fig. 6)

$$P_{v_1, v_4}, \quad P_{v_2, v_3}, \quad P_{v_2, v_4},$$

$n - k^{(3)} - 2$ edge paths of the form (cf. case 1)

$$P_{v_1, v, v_3}$$

$$\text{for } v \in V \setminus (V^{(3)} \cup \{v_2, v_4\})$$

and $n - k^{(2)} - k^{(4)} - 2$ edge paths of the form

$$P_{v_2, v^*, v_4}$$

for $v^* \in V \setminus (V^{(2)} \cup V^{(4)} \cup \{v_1, v_3\})$.

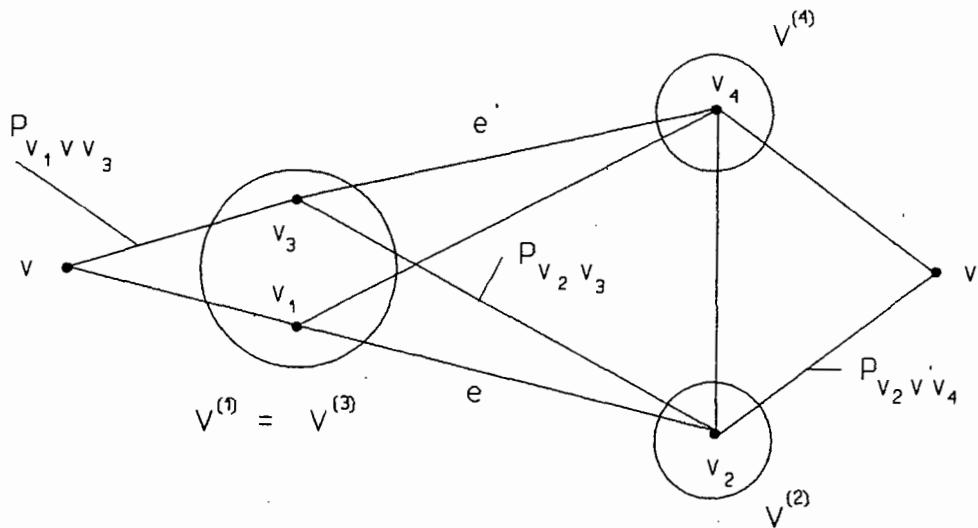


Fig. 6: Edge paths in the case 2a

Similarly to case 1 we furthermore obtain $k^{(2)} - 1$ edge paths of the form

$$P_0^{(t)} := (\{v_1, v_2\}, \{v_2, w_t'\}, \{w_t, w_t'\} \{w_t, v_4\} \{v_3, v_4\})$$

for $t = 1, \dots, k^{(2)} - 1$,

where $w_1, \dots, w_{k^{(2)} - 1}$ and $w'_1, \dots, w'_{k^{(4)} - 1}$ denote the elements of $V^{(2)} \setminus \{v_2\}$ and $V^{(4)} \setminus \{v_4\}$, respectively.

b) $v_1 = v_3$

We obtain the edge paths

$$(\{v_1, v_2\}, \{v_3, v_4\})$$

and P_{v_2, v_4} (cf. Fig. 7). Moreover, there are $n - k^{(3)} - 2$

edge paths of the form

$$P_{v_1, v, v_3} := (\{v_1, v_2\}, \{v_1, v\}, \{v_3, v_4\})$$

$$\text{for } v \in V \setminus (V^{(3)} \cup \{v_2, v_4\})$$

and $n - k^{(2)} - k^{(4)} - 1$ edge paths of the form

$$P_{v_2, v^*, v_4}$$

$$\text{for } v^* \in V \setminus (V^{(2)} \cup V^{(4)} \cup \{v_1\}) .$$

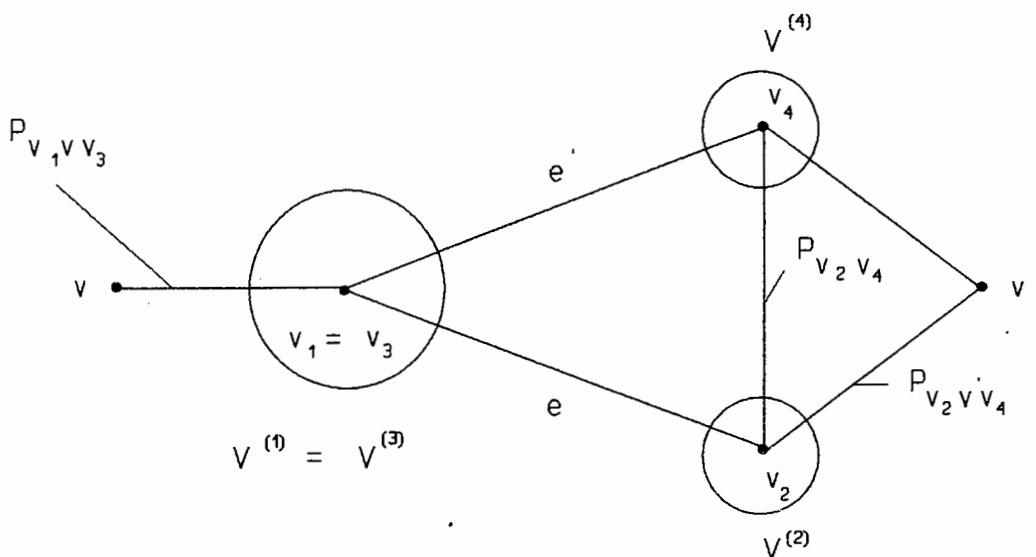


Fig. 7: Edge paths in the case 2b

Finally, we obtain $k^{(2)} - 1$ edge paths of the form

$$P_o^{(t)}$$

for $t = 1, \dots, k^{(2)} - 1$

(cf. case 2a).

Case 3:

$$V^{(1)} = V^{(3)} ; \quad V^{(2)} = V^{(4)}.$$

a) $v_1 \neq v_3$, $v_2 \neq v_4$

We obtain the edge paths (cf. Fig. 8)

$$P_{v_1, v_4} \text{ and } P_{v_2, v_3},$$

$n = k^{(3)} - 2$ edge paths of the form

$$P_{v_1, v, v_3}$$

for $v \in V \setminus (V^{(3)} \cup \{v_2, v_4\})$

and $n = k^{(4)} - 2$ edge paths of the form

$$P_{v_2, v, v_4}$$

for $v \in V \setminus (V^{(2)} \cup \{v_1, v_3\})$.

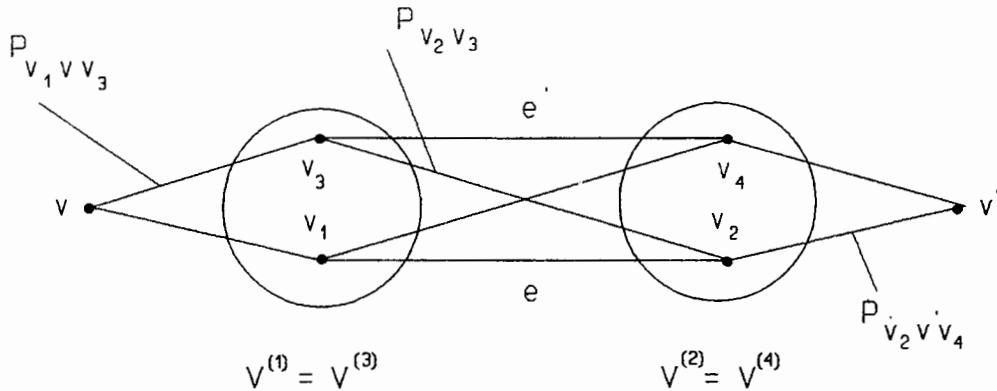


Fig. 8: Edge paths in case 3a

b) $v_1 = v_3$, $v_2 \neq v_4$

We obtain the edge path (cf. Fig 9)

$$(\{v_1, v_2\}, \{v_3, v_4\}) ,$$

$n - k^{(3)} - 2$ edge paths of the form (cf. case 2b)

$$P_{v_1, v, v_3}$$

for $v \in V \setminus (V^{(3)} \cup \{v_2, v_4\})$

and $n - k^{(4)} - 1$ edge paths of the form

$$P_{v_2, v', v_4}$$

for $v' \in V \setminus (V^{(2)} \cup \{v_1\})$. ■

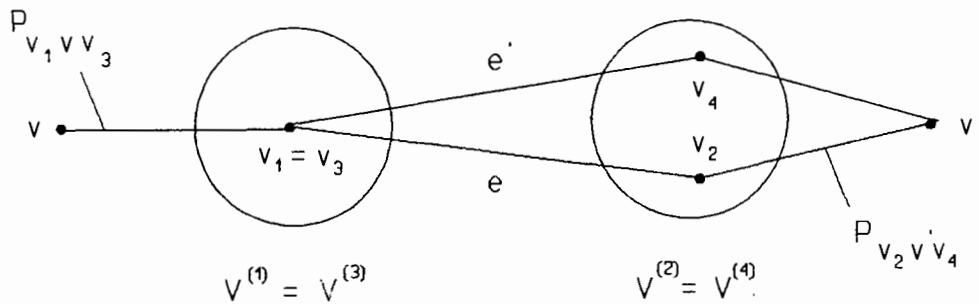


Fig. 9: Edge paths in case 3b

Lemma 2.1 implies the following result.

Theorem 2.3:

Let $G = K(p_1, \dots, p_r)$ denote a complete r -partite graph with $r \geq 2$, $p_1 \leq \dots \leq p_r$ and $n = \sum p_i$.

Then the connectivities κ' and λ' of the line graph $L(G)$ satisfy

$$\kappa' = \delta' = 2n - 2 - p_{r-1} - p_r$$

and

$$\lambda' = \delta' = 2n - 2 - p_{r-1} - p_r,$$

where δ' denotes the minimum degree of $L(G)$.

In order to prove only the second formula, we could also make use of the theorem in [9, p. 308].

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