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Abstract

We slightly modify the semantics of Moss' and Parikh's topological modal language ([MP]). This enables us to study the topological modal theory of further classes of subset spaces. Subsequently we deal in particular with spaces where every chain of opens is finite. We axiomatize the logic of those *quasi-finite* spaces, and prove *soundness* und semantical *completeness* of the proposed set of axioms. Moreover, it turns out that one can *decide* whether a given formula is a theorem of that logic. We also handle a somewhat wider class of subset spaces, which satisfy a more general finite–descent–condition on the set of opens.

1 Introduction

A few years ago, Moss and Parikh introduced a new logical framework for reasoning about knowledge ([MP]). Its syntactical part appears as a *bimodal language*. But the semantics is quite different from the usual interpretation of bimodal formulae in corresponding frames. Instead of it, the underlying semantical domains are so–called *subset–frames* $(X, \mathcal{O})$, where X is a non–empty set, and $\mathcal{O}$ is a set of subsets of X. The elements of $\mathcal{O}$ are called *opens*, and indeed *topologic*, i.e. the logic of subset frames where $\mathcal{O}$ is actually a topology on X, was soon an object of intensive study ([Geo 1], [DMP]).

How are the two modal operators - they will be denoted by K and $\Box$ respectively - interpreted in subset–frames? Well, "$\Box$" captures the *shrinking of an open*, while "K" captures *varying over an element of* $\mathcal{O}$. - How is this related to reasoning about knowledge? In its simplest form, knowledge representation of agents (e.g. processors in distributed systems) is modeled by the usual modal system $S5$. But herein only a *static* description is possible. More than that, to represent the dynamic process of *knowledge acquisition* is desirable by chance. *Topological modal logic* as proposed by Moss and Parikh offers (besides e.g. temporal approaches) an adequate opportunity to it. This becomes clear as soon as one notes that *reducing the number of thinkable alternatives increases knowledge.* Hence gaining more and more knowledge is in some sense a more and more accurate *approximation.* Since topological concepts are the right instrument to model approximations, we are rather naturally led to them within the posed problem.

As already indicated above, the approximation steps are managed by the $\Box$–operator. Thus $\Box$ models *effort* (whereas K models - as usual - *knowledge*), and that in a *non–deterministic* way: the outcome of effort is not known in advance; this is captured in the formal semantics by admitting a complete set of refinements of a given open. As the effort–operator's issue is a descent w.r.t. a system of sets, its modality is naturally $S4$–like.

At this point we take up a somewhat different view in comparison with [MP] . We assume that effort is in any case provided with *success.*

Thus $\Box$ is no longer $S4$– but $K4$–like in the present note. This is motivated first of all from a computational background: investment of computational resources certainly results in a better knowledge of the computed object (if the program is correct).

In the following we study two logics (one stronger than the other) which model situations where only *finitely often* effort can be successfully applied. The corresponding subset frames fulfil a *finite–descent–condition on the set of opens* respectively.

After introducing the logical language, we first give an *axiomatization* of the weaker system and prove its *soundness* and *completeness* w.r.t. the appropriate class of frames. Moreover, *decidability* of the logic can be proved as well. Proofs use to a great extent complicated constructions from [DMP]. Following up, the same results are obtained for the stronger system. But contrary to the weaker one, it satisfies the well-known *finite model property*: every formula not derivable in the system is falsified already in a finite model of the axioms. This latter property is proved via *tree–like spaces* considered recently by Georgatos [Geo 2], although we cannot apply Georgatos' results because of the modified semantics.

2 The Logical Language

For the present in accordance with [DMP], we introduce a language, called *topological modal logic* (TML), in the following way. The *syntax* of TML is based upon a suitable alphabet containing in particular symbols in order to define the set PV of *propositional variables.* The set TMF of *TML-formulae* is then defined recursively by the following clauses:

- $PV \cup \{\top\} \subseteq TMF$
- $\alpha, \beta \in TMF \Longrightarrow \neg\,\alpha, K\alpha, \Box\alpha, (\alpha \wedge \beta) \in TMF$
- no other strings belong to TMF

We omit brackets whenever possible, and use the following abbreviations (besides the usual ones from sentential logic): $L\alpha$ for $\neg K \neg\,\alpha$, $\Diamond\alpha$ for $\neg\Box\neg\,\alpha$.

The *semantical domains* of TML are generally triples $(X, \mathcal{O}, \sigma)$, where X is a non–empty set, $\mathcal{O}$ is a set of non–empty subsets of X, (possibly specified further), and $\sigma : PV \times X \longrightarrow \{0,1\}$ is a mapping (called X*-valuation*). The pair $\mathcal{S} = (X, \mathcal{O})$ is subsequently refered to as a *subset frame.* The elements of $\mathcal{O}$ sometimes are called *opens* (*of* $\mathcal{S}$). In this note we concentrate on two special kinds of subset frames, which we call (*weakly*) *quasi–finite.*

2.1 Definition

1. Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame. $\mathcal{S}$ is called (*weakly*) *quasi–finite*, iff every (desending) $\subset$–chain in $\mathcal{O}$ (having non–empty intersection) is finite (" $\subset$ " means *proper* inclusion).
2. Let $(X, \mathcal{O})$ be a *weakly* (quasi–finite) subset frame, and σ be an X–valuation. Then $\mathcal{M} = (X, \mathcal{O}, \sigma)$ is called a *weakly* (*quasi–finite*) *subset space* or a a *weakly* (*quasi–finite*) *model* (*based on* $(X, \mathcal{O})$).

We now define *validity* of TML–formulae in models based on subset frames. As motivated in the introduction, our definition differs slightly from the usual one in [MP], [DMP], [Geo 1], and [Geo 2].

2.1 Definition *Semantics of TML*

Let $\mathcal{M} = (X, \mathcal{O}, \sigma)$ be a subset–space model.

1. $X \otimes \mathcal{O} := \{(x, U) \mid U \in \mathcal{O}, x \in U\}$ is the set of *neighborhood situations* (of the underlying subset frame).
2. Now *validity* of a TML–formula *in model* $\mathcal{M}$ *at a neighborhood situation* x, U (brackets are omitted) is defined by recursion:

$$\begin{array}{lcl} x, U \models_{\mathcal{M}} \top & & \\ x, U \models_{\mathcal{M}} A & :\Longleftrightarrow & \sigma(x, A) = 1 \\ x, U \models_{\mathcal{M}} \neg\, \alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\ x, U \models_{\mathcal{M}} K\alpha & :\Longleftrightarrow & (\forall y \in U)\, y, U \models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & (\forall V \subset U)(x \in V \Longrightarrow x, V \models_{\mathcal{M}} \alpha) \end{array}$$

   for all $A \in PV$ and all formulae $\alpha, \beta \in TMF$.
3. Formula $\alpha \in TMF$ *holds in* $\mathcal{M}$ (denoted by $\models_{\mathcal{M}} \alpha$), iff it holds in $\mathcal{M}$ at every neighborhood situation.

If there is no ambiguity, we omit index $\mathcal{M}$ subsequently. - Examples of various subset frames and valid formulas (w.r.t. the usual semantics) are given in [DMP] and [Geo 2](e.g.).

3 The System MPG

We present, by a list of axioms and rules respectively, a first logical system **MPG**. Our aim is to show that the theorems of this system are exactly the TML–formulae which hold in every weakly quasi-finite model.

Axioms

1. All instances of propositional tautologies
2. $(A \rightarrow \Box A) \wedge (\neg A \rightarrow \Box \neg A)$
3. $K(\alpha \rightarrow \beta) \rightarrow (K\alpha \rightarrow K\beta)$
4. $K\alpha \rightarrow (\alpha \wedge KK\alpha)$
5. $L\alpha \rightarrow KL\alpha$
6. $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$
7. $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$
8. $K\Box\alpha \rightarrow \Box K\alpha$

for all $A \in PV$ and all $\alpha, \beta \in TMF$.

Rules

(1)	$\dfrac{\alpha \rightarrow \beta, \alpha}{\beta}$	(modus ponens)
(2)	$\dfrac{\alpha}{K\alpha}$	(K–necessitation)
(3)	$\dfrac{\alpha}{\Box\alpha}$	($\Box$–necessitation)

Some remarks on the axioms seem to be opportune. All but axiom (7) appear in the [MP]-list axiomatizing the subset space logic. Two schemes of this list are missing here:

$$\Box\alpha \rightarrow \Box\Box\alpha \text{ and } \Box\alpha \rightarrow \alpha$$

As it is well known, the first one can be derived from (6) and (7) with the aid of rules (1) and (3) (see e.g. [Fri], p. 190); the second scheme is canceled without compensation because of the modified semantics. (7) is the famous scheme W (also denoted as G) from ordinary modal logic.
Soundness of the axioms w.r.t. the intended structures can easily be established.

3.1 Proposition

Axioms (1) - (8) hold in every weakly quasi–finite model.

As to *completeness*, we first construct a subset space validating all of the above axioms, but falsifying a formula α which is not derivable in the system **MPG**. Fortunately, the complicated construction of [DMP], section 2.2, can be adapted for this purpose with only minor modifications. We only mention the variations here.

Clearly, the relation " $\xrightarrow{\diamond}$ " on the canonical model $\mathcal{M}_{MPG}$ of the logic (which exists because of the normality of the system w.r.t. each modality; see [Gol], §5) is no longer reflexive but only transitive (see [DMP], Prop. 2.1(2)). It turns out that reflexivity is not really needed. Now, in conditions (4), (L 4)(b), and (R 4)(b), " $\geq$ " has to be replaced by " $>$ ". This has also be done in the proof of Lemma 2.5 (where correspondingly " $\subset$ " is substituted for " $\subseteq$ "). The crucial change here should be compared with [DMP], Prop. 2.1(4), and is captured by the following Lemma.

3.2 Lemma

Let s be an element of the canonical model $\mathcal{M}_{MPG}$ of **MPG** such that $\Diamond\alpha \in s$. Then there exists a point t of $\mathcal{M}_{MPG}$ satisfying

$$s \xrightarrow{\Diamond} t \text{ and } \alpha \wedge \Box\neg\,\alpha \in t$$

Proof:
One shows that $\{\beta \mid \Box\beta \in s\} \cup \{\alpha \wedge \Box\neg\,\alpha\}$ is consistent. This can be achieved by assuming inconsistency towards a contradiction with the aid of axiom (7).
□

In the ongoing construction, Lemma 3.2 has to be used instead of the above mentioned proposition. We then have the following theorem.

3.3 Theorem

Let $\alpha \in TMF$ be a formula not derivable in the system **MPG**. Then there exists a subset space $\mathcal{X} = (X, \mathcal{O}^{\mathcal{X}}, \sigma)$ and a point $x \in X$ such that

- all axioms of **MPG** hold in $\mathcal{X}$, and
- $x, X \not\models \alpha$.

To turn the theorem to good account we must have a somewhat closer look at the [DMP]–construction. Actually, $\mathcal{O}^{\mathcal{X}}$ is obtained via an order–reversing injection i from a certain partially ordered set $(P, \leq)$ with least element $\perp$ into the set of non–empty subsets of X such that $i(\perp) = X$ and $\{q \mid q \leq p\}$ is linearly ordered for every $p \in P$. To be more precise, (X, P, i) is the limit of a sequence (X_n, P_n, i_n) such that for all $n \in \mathbb{N}$

- $X_n \subseteq X_{n+1}$,
- P_{n+1} is an (*end*) extension of P_n,
- $i_{n+1}(p) \cap X_n = i_n(p)$ for all $p \in P_n$.

Moreover, to every neighborhood situation $y, i(p)$ an element $t(y,p)$ of the canonical model $\mathcal{M}_{MPG}$ is associated during the construction, and every "existential" formula $\Diamond\beta$ resp. $L\beta$ corresponding to $y, i(p)$ in this way is eventually "realized". Finally, for all $\gamma \in TMF$, $\gamma \in t(y,p)$ iff $y, i(p) \models \gamma$. - After these preliminary remarks we can state the announced completeness theorem.

3.4 Theorem

> A formula $\alpha \in TMF$ is derivable in the system **MPG**, iff α holds in all weakly quasi–finite subset spaces.

Proof:
The "only if"–part is an immediate consequence of Proposition 3.1. - Now let α be not **MPG**–derivable. Consider the model $\mathcal{X}$ and the neighborhood situation x, X from Theorem 3.3.
The model we are looking for has carrier X and X–valuation σ (as $\mathcal{X}$ has). Its set of opens $\mathcal{O}$ is constructed inductively in stages with the aid of P:

stage 0: Set $\overline{P}_0 := \{\perp\} \subseteq P$.

stage $n+1$: For every $p \in \overline{P}_n$, every subformula $\Box\beta$ of α, and every $y \in i(p)$ such that $\Diamond\neg\beta \in t(y,p)$, choose an element $q \in P$, $q > p$, satisfying $\neg\beta \wedge \Box\beta \in t(y,q)$.
Let $\overline{P}_{n+1}$ the set of all those q.

Note that the existence of q is guaranteed through the modified [DMP]–

construction. - Now set $\overline{P} := \bigcup_{n \in \mathbb{N}} \overline{P}_n$, and let $\mathcal{O} := i(\overline{P})$. Furthermore, let $\mathcal{M} := (X, \mathcal{O}, \sigma)$.

Then $\mathcal{M}$ is weakly quasi–finite. This holds because there is only a finite number of subformulas $\Box\beta$ of α, and, if $p \in \overline{P}_n$ and $\Diamond\neg\beta \in t(y,p)$, then - because of $\neg\beta \wedge \Box\beta \in t(y,q)$ - $\Diamond\neg\beta$ cannot be a member of any $t(y,r)$, where $r > p$ is chosen in a step $k > n$.

We now prove by induction on the structure of formulae:

For all $\beta \in TMF$ and for all neighborhood situations y, U of $(X, \mathcal{O})$:
$[\beta \in sf(\alpha) \Longrightarrow (y, U \models_{\mathcal{X}} \beta \iff y, U \models_{\mathcal{M}} \beta)]$

(where $sf(\alpha)$ denotes the set of subformulas of α).

The cases $\beta =$ "$\top$" and β a propositional variable are clear from the definitions of $\mathcal{M}$ and " $\models$ ". If $\beta =$ "$\neg\gamma$", $\alpha =$ "$\gamma \wedge \delta$", or $\beta =$ "$K\gamma$", the induction hypothesis directly applies. The implication " $\Longrightarrow$ " in case $\beta =$ "$\Box\gamma$" follows easily from the induction hypothesis, since $\mathcal{O} \subseteq \mathcal{O}^{\mathcal{X}}$. In order to prove the reverse direction let y, U be a neighborhood situation of $(X, \mathcal{O})$ such that

$$y, U \not\models_{\mathcal{X}} \Box\beta.$$

($\Box\beta$ a subformula of α). By construction of $\mathcal{M}$, there is some $n \in \mathbb{N}$ and some $p \in \overline{P}_n \subset P$ such that $U = i(p)$. Since

$$y, U \not\models_{\mathcal{X}} \Box\beta \Longrightarrow y, U \models_{\mathcal{X}} \Diamond\neg\beta,$$

we get $\Diamond\neg\beta \in t(y,p)$. But in step $n+1$ of the above construction a $q \in P$, $q > p$, was chosen satisfying $\neg\beta \wedge \Box\beta \in t(y,q)$. Moreover, $V := i(q) \in \mathcal{O}$, and $V \subset U$ because of $q > p$. Thus we obtain

$$y, V \not\models_{\mathcal{X}} \beta.$$

By induction hypothesis,

$$y, V \not\models_{\mathcal{M}} \beta.$$

Consequently,

$$y, U \not\models_{\mathcal{M}} \Box\beta.$$

This ends the induction.
Since $x, X \not\models_{\mathcal{X}} \alpha$ by Theorem 3.3, the just proved assertion yields $x, X \not\models_{\mathcal{M}} \alpha$, as desired.
$\square$

4 Decidability

In this section we show that the set of formulas derivable in the system **MPG** is decidable. Again, we can go along the lines of [DMP] (section 2.3). We first introduce certain bimodal frames called **MPG**–*frames*.

4.1 Definition

1. Let $\mathcal{F} := (W, R, S)$ be a bimodal frame (i.e. W is a non–empty set and R, S are binary relations on W). Then $\mathcal{F}$ is called **MPG**–*frame*, iff
   - R is an equivalence relation on W;
   - S is irreflexive and transitive;
   - $(\forall s, t, u \in W)((s,t) \in R \wedge (t,u) \in S \Longrightarrow (\exists v \in W)[(s,v) \in S \wedge (v,u) \in R]$;
   - every ascending S–chain is finite.
2. A model $\mathcal{M} := (W, R, S, \sigma)$ based on an **MPG**–frame (W, R, S) is called an **MPG**–*model*, iff

   $$(\forall s, t \in W)(\forall A \in PV)((s,t) \in S \Longrightarrow [\sigma(A,s) = 1 \iff \sigma(A,t) = 1].$$

It is not difficult to see that **MPG** is sound and complete w.r.t. the just defined structures.

4.2 Proposition

> Every **MPG**–derivable formula holds in every **MPG**–model. Conversely, every $\alpha \in TMF$ which is not **MPG**–derivable is falsified in some **MPG**–model at some point.

Proof:
Every quasi–finite model $\mathcal{M} := (X, \mathcal{O}, \sigma)$ gives rise to an **MPG**–model $\tilde{\mathcal{M}} := (W, R, S, \tilde{\sigma})$ in the following way:

- $W := X \otimes \mathcal{O}$
- $((x, U), (y, V)) \in R :\iff U = V$
- $((x, U), (y, V)) \in S :\iff x = y \wedge V \subset U$
- $\tilde{\sigma}(A, (x, U)) = 1 :\iff \sigma(A, x) = 1$

An easy induction shows that for all $\alpha \in TMF$ the following holds:

$$(\forall x, U \in W)(x, U \models_{\mathcal{M}} \alpha \iff \tilde{\mathcal{M}} \models \alpha[x, U])$$

(where on the right–hand side usual (multi–modal) satisfaction (see [Gol], §5) is denoted).
Now the second assertion follows with the aid of Theorem 3.4. The first one is obvious.
□

Next we will show that every formula which holds in the canonical model $\mathcal{M}_{MPG}$ at some point is also satisfied in a *finite* **MPG**–model. Let $\alpha \in TMF$ satisfy

$$\mathcal{M}_{MPG} \models \alpha[s]$$

for some point s of $\mathcal{M}_{MPG}$.

Let Γ be the set of all subformulae of α joined with the set of all negated subformulae of α, $\tilde{\Gamma} := \Gamma \cup \{\beta \mid \beta$ is a finite conjunction of distinct elements of $\Gamma\}$, and $\Delta := \tilde{\Gamma} \cup \{L\beta \mid \beta \in \tilde{\Gamma}\}$. We then have the following filtration–lemma. (The reader not familiar with with filtrations should consult [Gol], §§4,5.)

4.3 Lemma

> Let $\mathcal{M} := (W, R, S, \sigma)$ be a Δ–filtration of $\mathcal{M}_{MPG}$ such that R and S are the minimal filtrations of the respective accessibility–relations in $\mathcal{M}_{MPG}$. Then $\mathcal{M}$ satisfies all **MPG**–model properties up to - possibly - irreflexivity.

Proof:

The proof is not simple. We need not present it here because its similarity with the [DMP]–proof. Especially, the semantic argument used to prove transitivity of S is applicable in our case, too. That is to say, we argue with completeness of the considered systems w.r.t. **MPG**–models stated in Proposition 4.2. (To be more precise, one profits by the fact that every consistent set of formulas containing all axioms holds in some MPG–model; note that $J \oplus_\Delta K$ - occuring in the proof of [DMP], Lemma 2.10 - is indeed an **MPG**–frame.)
□

As $\mathcal{M} := (W, R, S, \sigma)$ is a filtration of $\mathcal{M}_{MPG}$, W is finite ($\sharp W \leq 2^{\sharp\Delta}$) and consists of certain equivalence classes $\overline{s}$ of points s from the canonical model. Moreover, for all $\beta \in \Delta$ and every s we have

$$\mathcal{M}_{MPG} \models \beta[s] \iff \mathcal{M} \models \beta[\overline{s}].$$

Thus α is satisfied in the finite model $\mathcal{M}$. Unfortunately, $\mathcal{M}$ is not of the type we are looking for. But self–referential connections may simply be "forgotten".

4.4 Lemma

Let $\mathcal{M}$ be as above. Let $\mathcal{M}' := (W, R, S', \sigma)$, where $S' = S \setminus Diag(W)$. Then

$$(\forall \beta \in \Delta)(\forall v \in W)(\mathcal{M} \models \beta[v] \iff \mathcal{M}' \models \beta[v]).$$

Proof:
The induction is trivial except for the " $\Longleftarrow$ "–direction in the $\beta = \Box\gamma$–case. So let $v \in W$ such that $\mathcal{M} \not\models \Box\gamma[v]$. Then v is the class of some point s of the canonical model, and we have $\mathcal{M}_{MPG} \not\models \Box\gamma[s]$. Because of axiom (7) we have

$$\mathcal{M}_{MPG} \not\models \Box(\Box\gamma \rightarrow \gamma)[s].$$

Hence there exists a point t of $\mathcal{M}_{MPG}$, which is a " $\xrightarrow{\Diamond}$ " – successor of s, such that

$$\mathcal{M}_{MPG} \models (\Box\gamma \wedge \neg\gamma)[t].$$

Let u be the equivalence class of t. Then

$$\mathcal{M} \models (\Box\gamma \wedge \neg\gamma)[u] \text{ and } (v, u) \in S.$$

Since $\mathcal{M} \models \Box\gamma[u]$ we have $u \neq v$, and since $\mathcal{M} \models \neg\gamma[u]$ we obtain $\mathcal{M}' \models \neg\gamma[u]$ by induction hypothesis. Consequently, $\mathcal{M}' \not\models \Box\gamma[v]$.
$\Box$

Combining the results of this section we get the following theorem.

4.5 Theorem

The set $T := \{\alpha \in TMF \mid \alpha$ holds in every weakly quasi–finite model$\}$ is decidable.

Proof:

Let $\alpha \in TMF$ be given. Form Δ dependent on $\neg\,\alpha$ and let $n := 2^{\sharp\Delta}$. Check for all **MPG**–models $\mathcal{M} := (W, R, S, \sigma)$ such that $\sharp W \leq n$ whether $\mathcal{M} \models \neg\,\alpha[v]$ for some $v \in W$. If this is not the case, α is in T; otherwise not.

Correctness of this algorithm is proved as follows: if $\alpha \in T$, α clearly holds in every model considered above. Otherwise α is not **MPG**–derivable by Theorem 3.4, hence falsified in $\mathcal{M}_{MPG}$ at some point. Lemmata 4.3 and 4.4 yield that there exists a falsifying **MPG**–model of cardinality $\leq n$.

□

5 Concerning The FMP

In this final section we show first by giving a counterexample that the logic of weakly quasi–finite spaces does *not* have the *finite model property* (FMP).

5.1 Theorem

The logic of weakly quasi–finite spaces is lacking the FMP.

Proof:
Let $X := \mathbb{N}$. For $i \in \mathbb{N}$ let $U_i := \mathbb{N} \setminus \{0, \dots, i\}$, and define $\mathcal{O} := \{U_i \mid i \in \mathbb{N}\}$. Then $(X, \mathcal{O})$ is a weakly quasi–finite subset frame. Define an X–valuation σ by

$$\sigma(A, j) := 1$$

for all $A \in PV$ and $j \in \mathbb{N}$. Then the following formula holds in $\mathcal{M} := (X, \mathcal{O}, \sigma)$:

$$LA \wedge K(A \rightarrow L\Diamond A) \wedge K\Box(A \rightarrow L\Diamond A), \text{ where } A \in PV.$$

Clearly, this formula (call it α) cannot hold at any neighborhood situation of some finite subset frame. It follows that $\neg\,\alpha$ holds in every finite subset space. This implies (together with $\models_{\mathcal{M}} \alpha$) the lack of the FMP.
$\Box$

So far we have mainly dealt with weakly quasi–finite spaces. But some of the previously applied methods are also useful for studying the logic of quasi–finite spaces. Here is its axiomatization: Replace scheme (7) of the above list (see section 3) by

(9) $\Box(K\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha \quad (\alpha \in TMF)$.

The reader will notice that (9) is stronger then (7). - Let **QFS** be the resulting logical system. We then have the following theorem.

5.2 Theorem

> **QFS** gives a sound and complete axiomatization of the class of quasi–finite spaces.

Proof:

As to soundness, only (9) has yet to be considered. Validity in all quasi–finite models is easily established. - Completeness is proved analogously to that of **MPG**: See the proof of 3.4. But now (9) in its equivalent form

$$\Diamond\neg\,\alpha \rightarrow \Diamond(K\Box\alpha \wedge \neg\,\alpha)$$

has to be applied instead of (7). Clearly, the construction procedure then stops after a finite number of stages. Hence the resulting model is quasi–finite.

$\Box$

Note that the model falsifying a non **QFS**–derivable α is not only quasi–finite, but satisfies the stronger requirement that every $\subset$–chain in $\mathcal{O}$ is of bounded length (dependent on α). – Contrary to **MPG**, the system **QFS** satisfies the FMP.

5.3 Theorem

> Every formula α not **QFS**–derivable is falsified in a finite subset space $\mathcal{M} = (X, \mathcal{O}, \sigma)$. Moreover, the cardinality of X depends on the size of α.

Before proving the theorem, let us state an immediate corollary.

5.4 Corollary

> The set $\widetilde{T} := \{\alpha \in TMF \mid \alpha$ holds in every quasi–finite model$\}$ is decidable.

This result could also have been obtained using the methods of section 4. Indeed, if one changes Definition 4.1(1) in that *every* S–chain is finite (resulting in the notion of a **QFS**–*frame*), one can take over the argumentation from there. But, clearly, the FMP is a stronger attribute of the system.

Proof of Theorem 5.3:

Let $\alpha \in TMF$ be not **QFS**–derivable. By (the proof of) theorem 5.2, there exists a quasi–finite model $\mathcal{M} = (X, \mathcal{O}, \sigma)$ and a point $x \in X$ such that $x, X \not\models \alpha$. We shall now associate with $\mathcal{M}$ a so–called *tree–like model* (see [Geo 2]) falsifying α as well.

5.5 Definition

A subset space $\mathcal{M}' = (X', \mathcal{O}', \sigma')$ is called a *tree–like model*, iff for all $U, V \in \mathcal{O}'$

$$U \subseteq V \text{ or } V \subseteq U \text{ or } U \cap V = \emptyset.$$

In [Geo 2] it was proved that every TMF–formula β which is satisfiable in some tree–like model is also satisfiable in some finite tree–like model (Theorem 31), where the number of points is a function of the complexity of α.

5.6 Lemma

Let $\mathcal{M} = (X, \mathcal{O}, \sigma)$ be as above. Then there exists a tree-like model $\mathcal{M}' = (X', \mathcal{O}', \sigma')$ and a surjection $\varphi : X' \to X$ which induces an inclusion-preserving bijection from $\mathcal{O}'$ onto $\mathcal{O}$ such that

$$(\forall \beta \in TMF)(y, U \models_{\mathcal{M}'} \beta \iff \varphi(y), \varphi(U) \models_{\mathcal{M}} \beta$$

for all neighborhood situations y, U of $(X', \mathcal{O}')$.

Proof:

Let $Y := \{(y; y, U) \mid y \in X;\ y, U \in X \otimes \mathcal{O}\}$. We shall construct the desired model in a finite number of steps such that $X' \subseteq Y$.

stage 0: Let I_0 be the set of minimal opens of $\mathcal{O}$.
Let $V' := \{(y; y, V) \mid y, V \in X \otimes \mathcal{O}\}$,
and define $\mathcal{O}'_0 := \{V' \mid V \in I_0\}$.

stage $n+1$: Let $\mathcal{O}_n$ be already constructed. Let I_{n+1} be the set of minimal opens of $\mathcal{O}$ not yet processed. For every $U \in I_{n+1}$ let $U' := \bigcup\limits_{V \subset U} V' \cup \{(y; y, U) \mid y, U \in X \otimes \mathcal{O};\ y \notin \bigcup\limits_{V \subset U} V\}$,
and define $\mathcal{O}_{n+1} := \{U' \mid U \in I_{n+1}\}$.

Since every $\subset$–chain in $\mathcal{M}$ is of bounded length (see the remark after 5.2), the procedure is finally finished; say after stage k. Now let

$$\mathcal{O}' := \bigcup_{n=0,\ldots,k} O_n, \text{ and } X' := \bigcup \mathcal{O}'.$$

It remains to define σ'. But its definition is canonical:

$$\sigma'(A, (y; y, U)) := \sigma(y)$$

for all $A \in PV$ and all $(y; y, U) \in X'$. Now, as one can see inductively, $\mathcal{M}' = (X', \mathcal{O}', \sigma')$ is a tree-like model.

The mapping φ is likewise defined in a natural manner:

$$\varphi(y; y, U) := y \text{ for all } y, U \in X \otimes \mathcal{O}.$$

Note that φ is indeed surjective, since $X \in \mathcal{O}$. Clearly, $\varphi(U') = U$ and $V \subseteq U$ iff $V' \subseteq U'$ for all $U, V \in \mathcal{O}$.

So far we have defined $\mathcal{M}'$ and φ. The final assertion of the lemma is then proved by a simple induction on the structure of β which is not carried out here.

$\square$ (5.6)

Thus our original α is falsified in a tree–like model. Unfortunately we cannot use Georgatos' above mentioned result now, since we changed the semantics.

Instead of it we proceed follows (see the proof of 3.4 and 5.2 resp.):
Let $\mathcal{M} = (X, \mathcal{O}, \sigma)$ be the tree–like model falsifying α, which was constructed in lemma 5.6 in dependence on the corresponding model from the proof of 5.2. Then, according to the "distance" of an open V from X w.r.t. proper reverse inclusion, a partition of $\mathcal{O}$ is induced:

$$\mathcal{O} = \bigcup_{i=0,\ldots,k} \mathcal{O}_i$$

for a certain $k \in \mathbb{N}$, such that for all $x \in X$ there is at most one $V_x^i \in \mathcal{O}_i$ satisfying $x \in V_x^i$ ($\mathcal{O}_0 = \{X\}$); if $V \in \mathcal{O}_i$, we call i the *level* of V.

Now define inductively finite sets $\mathcal{O}^i$ of opens as follows:

$i = 0$ $\qquad \mathcal{O}^i := \{X\}$.

$i = n + 1$: Let $V \in \mathcal{O}^n$ and $\square\beta \in sf(\alpha)$ Choose some $x \in X$ satisfying $x, V \models \Diamond\neg\beta$ (if such an x exists), and let $V^{n+1}_{x,\square\neg\beta} := V^{n+1}_x$.
Define $\mathcal{O}^{n+1} := \{V^{n+1}_{x,\square\beta} \mid \square\beta \in sf(\alpha), V^n_x$ exists, and $x, V^n_x \models \Diamond\neg\beta$ for some $x\}$

(Note that $x, V^{n+1}_{x,\square\beta} \models \neg\beta \wedge K\square\beta$ and that $\mathcal{O}^n \subseteq \mathcal{O}_n$ for every $0 \leq n \leq k$.)

Let

$$\tilde{\mathcal{O}} := \bigcup_{i=0,\ldots,k} \mathcal{O}^i.$$

Then $\tilde{\mathcal{O}}$ is finite. For each $U \in \tilde{\mathcal{O}}$ let

$$\overline{U} := U \setminus \bigcup_{V \subset U, V \in \tilde{\mathcal{O}}} V,$$

and define the following equivalence relation on the set of neighborhood situations x, V satisfying $x \in \overline{U}$ and $V \subset \overline{U}$:

$$x, V \sim_U y, W :\iff (\forall K\beta \in sf(\alpha))[x, V \models K\beta \iff y, W \models K\beta].$$

Clearly, there is only a finite number of equivalence classes, which we denote by $[x, V]_U$ and collect as a set $\mathcal{U}$.
Let an equivalence class $[x, V]_U \in \mathcal{U}$ be given. Then let

$$< x, V >_U := \{V \mid (\exists y)\; y, V \in [x, V]_U\}.$$

We now define inductively a partition $\widehat{\mathcal{U}}_j$ of $\overline{\mathcal{U}}_j := \{V \in \mathcal{O} \mid V \subset \overline{U}\} \cap \mathcal{O}_{i+j}$ for all $1 \leq j$ such that $i + j \leq k$, where i is the level of U.

$j = 1$: Let $\widehat{\mathcal{U}}_1 := \{< x, V >_U \cap \overline{\mathcal{U}}_1 \mid [x, V]_U \in \mathcal{U}\}$.

$j = n + 1$: Let $\widehat{\mathcal{U}}_n$ be already defined. Take any element $\mathcal{V} \in \widehat{\mathcal{U}}_n$. Consider the intersection
$\bigcup \mathcal{V} \cap \bigcup \overline{\mathcal{U}}_{n+1}$.
If this intersection is properly contained in $\bigcup \mathcal{V}$,
let $\mathcal{V}' := \{< x, V >_U \cap \overline{\mathcal{U}}_{n+1} \mid V \in \mathcal{V}\}$.
Otherwise choose a non-trivial partition $\mathcal{V}_1, \mathcal{V}_2$ of $\mathcal{V}$, and let
$\mathcal{V}_1' := \{< x, V >_U \cap \overline{\mathcal{U}}_{n+1} \mid V \in \mathcal{V}_1\}$ and
$\mathcal{V}_2' := \{< x, V >_U \cap \overline{\mathcal{U}}_{n+1} \mid V \in \mathcal{V}_2\}$.
Subsume all those $\mathcal{V}'$ and $\mathcal{V}_1', \mathcal{V}_2'$ resp. under the set $\widehat{\mathcal{U}}_{n+1}$.
(Note that a non-trivial partition $\mathcal{V}_1, \mathcal{V}_2$ exists whenever the the above considered intersection does not fulfil the proper-containment-condition.)

Now define $\widehat{\mathcal{O}} := \tilde{\mathcal{O}} \cup \{\bigcup \mathcal{V} \mid \mathcal{V} \in \widehat{\mathcal{U}}_j \text{ for some } 1 \leq j \leq k - i\}$.

Clearly, $\widehat{\mathcal{O}}$ is finite as well. Define a (surjective) function $\varphi : \mathcal{O} \to \widehat{\mathcal{O}}$

by

$$\varphi(V) := \begin{cases} V & \text{if } V \in \tilde{\mathcal{O}} \\ \bigcup(< x, V >_U \cap \overline{\mathcal{U}}_l) & \text{if } U \text{ is the minimal element of } \tilde{\mathcal{O}} \text{ containing } V, \\ & \text{and } l \text{ is the level of } V. \end{cases}$$

With these notations we have the following lemma.

5.7 Lemma

Let $\mathcal{M}$ be the above, and $\widehat{\mathcal{M}} := \{X, \hat{\mathcal{O}}, \sigma\}$. Then for all subformulas β of α, all $x \in X$ and every $V \in \hat{\mathcal{O}}$ the following holds:

$$x, V \models_{\mathcal{M}} \beta \iff x, \varphi(V) \models_{\widehat{\mathcal{M}}} \beta.$$

Proof:

The sentential–logical cases are evident.

$\underline{\beta = \Box\gamma}$:

The direction " $\Longrightarrow$ " is clear by the induction hypothesis because of the surjectivity of φ. As to the other direction, let $x, V \not\models_{\mathcal{M}} \Box\gamma$. Then there exists a $V' \subset V$ such that $x, V \not\models_{\mathcal{M}} \gamma$. By induction hypothesis, $x, \varphi(V) \not\models_{\widehat{\mathcal{M}}} \gamma$. As our above construction respects the levels of opens and leads in any case to proper containment, $x, \varphi(V) \not\models_{\widehat{\mathcal{M}}} \Box\gamma$.

$\underline{\beta = K\gamma}$:

In this case the assertion follows because $\sim_U$ was defined adequately, and the sets $\overline{\mathcal{U}}_j$ of partitions respect the equivalence classes.
$\Box$ (5.7)

No further difficulties arise in the course of the proof of Theorem 5.3 along Georgatos' outline ([Geo 2], Lemma 30 ff.).
□ (5.3)

6 Conclusions

We changed slightly the semantics of topological modal logic introduced by Moss and Parikh ([MP]): we considered *strict* inclusion instead of ordinary inclusion in modelling the box–operator. As a first application we treated the logics of *(weakly)–quasi–finite* spaces. We proved *completeness* of the proposed logical systems, and showed *decidability* of their respective set of theorems. Moreover, the finite model property does not hold for the logic of weakly quasi–finite spaces, whereas it holds if one restricts interpretation to quasi–finite spaces.
It seems to be interesting to study further logics based on the here presented modified semantics, especially in a computational context where one is confronted with *approximations* of objects rather than objects themselves.

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